Pricing and Hedging Calendar Spread Options on Agricultural Grain Commodities

by

Adam Schmitz, Zhiguang Wang, and Jung-Han Kimn

Suggested citation format:

Schmitz, A., Z. Wang, and J.-H. Kimn. 2013. “Pricing and Hedging Calendar Spread Options on Agricultural Grain Commodities.” Proceedings of the NCCC-134 Conference on Applied Commodity Price Analysis, Forecasting, and Market Risk Management. St. Louis, MO. [http://www.farmdoc.illinois.edu/nccc134].

Pricing and Hedging Calendar Spread Options on Agricultural Grain Commodities

Adam Schmitz, Zhiguang Wang, and Jung-Han Kimn[*]

Paper presented at the NCCC-134 Conference on Applied Commodity Price Analysis, Forecasting, and Market Risk Management

St. Louis, Missouri, April 22-23, 2013

[*] Schmitz (Adam.Schmitz@sdstate.edu) is a PhD candidate in the Department of Mathematics and Statistics at South Dakota State University. Wang (Zhiguang.Wang@sdstate.edu) and Kimn (Jung-Han.Kimn@sdstate.edu) are assistant professors in the Department of Economics and the Department of Mathematics and Statistics at South Dakota State University, respectively. The financial support from Agricultural Experiment Station at South Dakota State University (Project H363-10) and Stahly Scholar in Financial Economics are gratefully acknowledged by Wang. We appreciate the comments by Dwight Sanders (session chair), Wade Brorsen, Scott Irwin, Aaron Smith and other participants at the NCCC-134 conference.

Pricing and Hedging Calendar Spread Options on Agricultural Grain Commodities

Adam Schmitz*, Zhiguang Wang†, and Jung-Han Kimn‡

June 1, 2013

Abstract

The calendar spread options (CSOs) on agricultural commodities, most notably corn, soybeans and wheat, allow market participants to hedge the roll-over risk of futures contracts. Despite the interest from agricultural businesses, there is lack of both theoretical and empirical research on pricing and hedging performances of CSOs. We propose to price and hedge CSOs under geometric Brownian motion (GBM) and stochastic volatility (SV) models. We estimate the model parameters by using implied state-generalized method of moments (IS-GMM) and evaluate the in-sample and out-of-sample pricing and hedging performances. We find that the average pricing errors of the SV model are 0.79% for corn, 0.75% for soybeans and 1.2% for wheat; the pricing and hedging performance of the SV model are mostly superior to the benchmark GBM model, both in and out of sample, with only one exception where the out-of-sample hedging error for the GBM model for market makers is slightly better than the SV model.

1 Introduction

A calendar spread option (CSO) gives the owner the right but not the obligation to simultaneously buy and sell futures contracts with different expiration dates at a predetermined strike price. Agricultural CSOs on corn, soybeans and wheat began trading in the summer of 2009 on the Chicago Mercantile Exchange (CME).[1] The trading of CSOs has steadily increased to daily volumes of 512 contracts for corn, 546 for soybeans and 51 for wheat, as investors have become more aware of risk management benefits of these options. The most popular CSOs are the new crop-old crop options, establishing long and short positions on

*Ph.D. candidate in the Department Mathematics and Statistics at South Dakota State University, Brookings, SD 57007, USA

†Assistant Professor and Stahly Scholar in Financial Economics, Department of Economics at South Dakota State University, Brookings, SD 57007, USA

‡Assistant Professor in the Department Mathematics and Statistics at South Dakota State University, Brookings, SD 57007, USA

[1]The CME Group also offers Live Cattle and Lean Hogs CSOs which are settled in the same way as the grain CSOs. In this paper, we focus on grain CSOs due to their superior liquidity.

contracts for the new and old crops. The most actively traded new crop-old crop contracts for corn, soybeans and wheat are July-December, July-November, and December-July contracts, respectively, although other contract months are available.

The main motivation for trading CSOs is their ability to hedge the price uncertainty with the rolling over of contract. Depending on the hedger's positions in cash and futures, different options strategies can help mitigate risk exposure. Consumers of grain, such as ethanol producers that need to constantly fill their corn inventories, can buy a CSO put to minimize the risk of falling nearby or rising deferred futures prices. This long put position creates a floor for potential losses the producer could incur. On the other hand, suppliers, such as grain elevators that hold the cash crop and need to short futures, can buy a CSO call option. This position of short nearby futures and long the consecutive CSO call locks in the spread between the nearby and the deferred which again creates a floor for losses due to a narrowing of the spread. If the spread widens, the supplier has exposure to the upside allowing for profit.[2]

Academic research on the pricing and hedging of agricultural CSOs has yet to match agricultural businesses' interest in these products, despite a number of studies on generic spread options on financial assets, namely stocks. There is lack of theoretical and empirical research on agricultural and calendar spread options on futures (as opposed to cash assets). Past studies mainly aims to provide a pricing formula and to verify its accuracy through Monte Carlo simulation. Ravindran (1993) and Kirk and Aron (1995) proposes an approximation formula for pricing spread options under the Geometric Brownian Motion (GBM) framework. Carmona and Durrleman (2003) survey different theoretical and computational techniques for pricing spread options. Dempster and Hong (2002) and Hurd and Zhou (2013) use the Fourier inversion for option pricing and implement the Fast-Fourier Transform (FFT) to compute prices. Fang (2006), Lord, et al. (2008), and Leentvaar and Oosterlee (2008) modify the Fourier technique found in Carr and Madan (1999) to increase computational efficiency, especially for pricing multi-asset options.

We fill the gap in the literature of CSOs, particularly agricultural CSOs, by providing a theoretical pricing/hedging solution and calibrating the proposed model to real market data. We model the dynamics of futures prices using both the Geometric Brownnian Motion (GBM) and a mean-reverting stochastic volatility model (SV). There has been an increase in the need for market participants to understand and hedge risk associated with volatility in the agricultural futures markets. To fulfill the market's need for a volatility hedging product, the CME group and Chicago Board Options Exchange (CBOE) introduced the corn and soybean volatility indexes to the market in early 2011. In academia, stochastic volatility has become an indispensable feature of financial asset prices as lead by Heston (1993). Schwartz (1997) proposes three models using stochastic volatility model and stochastic convenience yield in order to model copper, oil and gold. Trolle and Schwartz (2009) study the effects of unspanned stochastic volatility on commodity derivatives based on the Heath, Jarrow,

[2]The CME group website provides a number of examples on how to utilize CSOs. Examples and mechanics of CSO can be found at
http://www.cmegroup.com/trading/agricultural/files/IntroductionCalendarSpreadOptionsGrainsOilseedProducts.pdf.

and Morton (1992) risk-neutral measure. Geman and Nguyen (2005) propose a two-factor stochastic volatility model to study soybean stocks and price volatility.

As a comparison, we consider both a two-factor Geometric Brownian Motion (GBM) model and a three-factor stochastic volatility (SV) model. We fit the CME data to the GBM model by minimizing least squared errors and to the SV model by employing an Implied-State Generalized Method of Moments (IS-GMM) proposed by Pan (2002). We find that the parameter estimates for both models are intuitive. The volatility of the nearby futures in both GBM models is higher than the volatility of the deferred, evidence of backwardation in the futures volatilities. This is likely due to the fact that any new information affects the front month futures more than the deferred futures. There is a nearly perfect correlation between the nearby and deferred futures returns in the GBM model for the three products. Although the correlation between the futures prices for the SV model is not as strong as for the GBM model, it is still relatively high. The volatility half-time ranging from three to five months largely coincides with the gap between the old and new crop months, which drives the seasonality of the trading and possibly the behavior of grain market volatility.

Furthermore, we find both models generate less than 5 cents of pricing error on average and with the SV model's error being even less than 1 cent. More specifically, the SV model produces the average percentage errors of 0.79% for corn, 0.75% for soybeans and 1.2% for wheat. However, the pricing and hedging performance of the SV model are mostly superior to the benchmark GBM model, both in and out of sample, with only one exception where the out-of-sample hedging error for the GBM model for market makers is slightly better than the SV model. The empirical results lend strong support for agribusinesses and the market makers to adopt the SV model for pricing and hedging grain CSOs. Another implication for the exchange (CME) is to employ the SV model, instead of the Black-Scholes-type GBM model for determining the settlement prices.

The remainder of the paper is organized as follows: Section 2 states our model and the propositions relevant to pricing the CSOs; Section 3 describes the data set and the estimation procedures; Section 4 presents the parameter estimation results and analyzes the in-sample and out-of-sample pricing errors and hedging capabilities; and Section 5 concludes this paper.

2 Models for Agricultural Calendar Spread Options

2.1 Geometric Brownian Motion vs. Stochastic Volatility Models

We first consider a correlated two-factor geometric Brownian motion (GBM) model as a benchmark for futures price dynamics. The nearby and deferred futures prices follow a log normal distribution with different volatilities that permit a simple term structure. The model is specified under the risk-neutral $\mathcal{Q}$ measure as follows:

$$
\begin{aligned}
d\ln F(t,T_1) &= (-\sigma_1^2/2)dt + \sigma_1 dW_1 && (1)\\
d\ln F(t,T_2) &= (-\sigma_2^2/2)dt + \sigma_2 dW_2 && (2)\\
E[dW_1, dW_2] &= \rho\, dt. && (3)
\end{aligned}
$$

Here, $F(t,T_1)$, $F(t,T_2)$, σ_1 and σ_2 represent the prices and their (return) volatilities of the nearby and deferred futures prices, respectively.

As a comparison, we augment the two-factor GBM model with a latent Heston-type stochastic volatility (SV) to obtain a three-factor SV model.[3] The well-known Heston (1993) model features a mean-reverting process with a square root function of volatility in the diffusion term. The three-factor SV model reads:

$$
\begin{aligned}
d\ln F(t,T_1) &= (-\sigma_1^2 v/2)dt + \sigma_1\sqrt{v}dW_1 && (4)\\
d\ln F(t,T_2) &= (-\sigma_2^2 v/2)dt + \sigma_2\sqrt{v}dW_2 && (5)\\
dv &= \kappa(\mu - v)dt + \sigma_v\sqrt{v}dW_v && (6)
\end{aligned}
$$

with μ being the long-run mean of stochastic variance v_t and κ its mean reversion rate. The correlation structure under the risk neutral measure $\mathcal{Q}$ are

$$
\begin{aligned}
E^{\mathcal{Q}}[dW_1, dW_2] &= \rho_{12}\, dt && (7)\\
E^{\mathcal{Q}}[dW_1, dW_v] &= \rho_{1v}\, dt && (8)\\
E^{\mathcal{Q}}[dW_2, dW_v] &= \rho_{2v}\, dt. && (9)
\end{aligned}
$$

2.2 Pricing Options on Futures Contracts

For a nearby futures contract expiring at T_1 and a deferred futures contract expiring at T_2, the CSO call's payoff at time t maturing at time T with strike K is $\max\{F(t,T_1)-F(t,T_2)-K, 0\}$. The CSO price at time t is given by

$$
\begin{aligned}
C_t(F_1, F_2; K, T) &:= \mathbb{E}^{\mathcal{Q}}\left[e^{-rT}[F_1 - F_2 - K]_+\right]\\
&= \int_\Omega\int_\Omega e^{-rT}\left(e^{f_2} - e^{f_1} - K\right) q_T(f_1, f_2)df_2 df_1 \qquad (10)
\end{aligned}
$$

where $f_i = log(F(t,T_i)) = log(F_i)$ for $i = 1, 2$ are log futures prices and $q_T(f_1, f_2)$ is the joint probability density of f_1 and f_2. Ω is the domain of (log) futures prices. The expectation can be integrated through the Gaussian quadrature for the GBM model. The more general and

[3]More sophisticated model features, such as seasonality and term structure of cost of carry, have been considered by Geman and Nguyen (2005), Back, Prokopczuk and Rudolf (2013), and Schmitz, Wang and Kimn (2013) among others. We do not include these additional features, considering the following tradeoff with a more complex model: (1) the lack of the closed-form solutions to the characteristic function and the pricing formula; and (2) the challenge in terms of fast, reliable and efficient parameter estimation. Our empirical results (pricing error as low as 1%) show that additional features would add little in modeling agricultural CSOs.

popular approach is to evaluate the probability via Fourier transform of the characteristic function, which admits a closed-form for both models considered in this research.

As for the first approach, Ravindran (1993) and Kirk and Aron (1995) provide an approximation based on the Black-Scholes formula for the GBM model. The Ravindran formula for the call spread option reads:[4]

$$\begin{aligned} C(F_1, F_2; K, T) &= E_{F_1}\left[C_{BS}(F^*, K^*, \sigma^*, T)|F_2\right] \\ &= \sum_i w_i n(x_i) C_{BS}(F_1^*, K^*, \sigma^*, T) \qquad (11) \\ F_1^* &= F_1 \exp(\rho\sigma_1\sqrt{T}x_i - \frac{1}{2}\sigma_1^2\rho^2 T) \\ K^* &= F_2 \exp(\sigma_2\sqrt{T}x_i - \frac{1}{2}\sigma_2^2 T) + K \\ \sigma^* &= \sigma_1\sqrt{1-\rho^2}. \end{aligned}$$

C_{BS} is the well-known Black-Scholes formula. w_i and x_i are the Gauss-Legendre quadrature weigts and abscissas in the range from -4 to +4. $n(.)$ is the normal density function. Other parameters are specified in Equations (1-3).

The Fourier-transformation-based approach has been widely adopted in the derivatives pricing literature: Carr and Madan (1999) for single-asset options, Dempster and Hong (2002), Hurd and Zhou (2013), Lord et al. (2008), Leentvaar and Oosterlee (2008) and Fang (2006) for multi-asset options. The last three follow the same approach utilizing the property of independent increments of any Lévy process. The evaluation of Equation (10) becomes a multi-dimensional convolution method that takes the following form:

$$\begin{aligned} C(F_1, F_2; K, T) &= \mathbb{E}^{\mathcal{Q}}\left[e^{-rT}[F(t,T_1) - F(t,T_2) - K]_+\right] \\ &= \exp(-rT)\mathcal{F}^{-1}\{\mathcal{F}\{C(F_1, F_2; K, T)\}\phi(-u_1, -u_2)\} \qquad (12) \end{aligned}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms, respectively. ϕ denotes the characteristic function of the frequency vector (q_1, q_2).

The characteristic function of log futures prices at expiration for the GBM model is given by

$$\begin{aligned} \phi_{gbm}(u_1, u_2) &= E^{\mathcal{Q}}[\exp(\mathrm{i}u_1 lnF_1(T) + \mathrm{i}u_2 lnF_2(T))] \\ &= \exp\Bigg[\mathrm{i}u_1 \ln F_1 + \mathrm{i}u_2 \ln F_2 - \frac{1}{2}\Big[\left(\sigma_1^2u_1^2 + \sigma_2^2u_2^2 + 2\rho\sigma_1u_1\sigma_2u_2\right) \\ &\quad + \mathrm{i}\left(\sigma_1^2u_1 + \sigma_2^2u_2\right)T\Big]\Bigg]. \qquad (13) \end{aligned}$$

[4] In the interest of space, we do not reproduce Kirk's approximation formula. These two formulae produce numerically very similar results.

The following proposition gives the closed-form characteristic function for the three-factor SV model.

Proposition 1. *For $u_1, u_2, s_1,$ and $s_2 \in \mathbb{R}$, $T \geq 0$, $v, \sigma_v, \sigma_1, \sigma_2 > 0$, and $\rho, \rho_1, \rho_2 \in [-1, 1]$, the characterestic function for the SV model is given by*

$$\begin{aligned}
\phi_{sv}(u_1, u_2) &= E^{\mathcal{Q}}[\exp(\mathrm{i}u_1 \log F_1(T) + \mathrm{i}u_2 \log F_2(T)] \\
&= \exp\left[\mathrm{i}u_1 lnF_1 + \mathrm{i}u_2 lnF_2 + \left(\frac{2\zeta(1-\exp^{-\theta T})}{2\theta - (\theta-\gamma)(1-\exp^{-\theta T})}\right) v(0)\right. \\
&\quad \left. -\frac{\kappa\mu}{\sigma_v^2}\left[2\log\left(\frac{2\theta - (\theta-\gamma)(1-\exp^{-\theta T})}{2\theta}\right) + (\theta-\gamma)T\right]\right]
\end{aligned}$$

with

$$\begin{aligned}
\zeta &= -\frac{1}{2}\left[\left(\sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 + 2\rho\sigma_1 u_1 \sigma_2 u_2\right) + \mathrm{i}\left(\sigma_1^2 u_1 + \sigma_2^2 u_2\right)\right] \\
\gamma &= \kappa - \mathrm{i}(\rho_1\sigma_1 u_1 + \rho_2\sigma_2 u_2)\sigma_v \\
\theta &= \sqrt{\gamma^2 - 2\sigma_v^2\zeta}.
\end{aligned}$$

Proof. See Appendix. □

The options value $C(.; K, T)$ is calculated using the Fast Fourier Transform (FFT) with the discretization size of the log price (f_1 and f_2) and frequency domains (u_1 and u_2) in the following:

$$\begin{aligned}
f_{j,k_j} &= f_{j,0} + k_j\Delta_j, \\
u_{j,k_j} &= u_{j,0} + n_j\lambda_j, \\
k_j, n_j &= 0, 1, ..., N-1,
\end{aligned}$$

$\forall j = 1, 2$ and with the Nyquist relation $\lambda_1 \cdot \Delta_1 = \lambda_2 \cdot \Delta_2 = \frac{2\pi}{N}$ to avoid aliasing.

The discretization of Equation (12) yields the call option value:

$$\begin{aligned}
C(F_1, F_2; K, T) &= \frac{exp(-rT)}{(2\pi)^2}\prod_{j=1}^{2}(-1)^{k_j}\mathcal{D}^{-1}\{\phi_{n_j}\mathcal{D}\left[C_k G_k\right]\} \qquad (14) \\
G_k &= \prod_{j=1}^{2} R_j(k_j)
\end{aligned}$$

with $R_j(k_j) = \frac{1}{2}$ for $k_j = 0, N-1$ and equal to 1 otherwise. The FFT computation is performed on Nvidia Graphical Processing Unit (GPU) to take advantage of its massively parallel structure. The put option value $P(F_1, F_2; K, T)$ can be easily obtained via the put-call parity $P(F_1, F_2; K, T) = C(F_2, F_1; -K, T)$.

3 Data and Methodology

3.1 Data

In this paper we focus on three most liquid grain CSOs, namely July-December Corn, July-November Soybeans and December-July wheat CSO contracts. This type of option is often referred to as the "old crop-new crop" option, allowing the buyer to hedge the price difference in a rollover of the new incoming crop yield. We filter the data by including only those calls that have a positive volume for the CSO and are the most near-the-money to ensure data quality. The trading dates range from January 7, 2010, to September 27, 2012, for corn, January 8, 2010, to May 29, 2012 for soybeans, and February 11, 2011 to September 24, 2012 for wheat. Interest rate data are obtained from the Federal Reserve St. Louis and are interpolated to match the maturity of CSOs. Table 1 presents the descriptive statistics for the years for the corresponding in-sample days for both nearby and deferred futures, and the calendar spread option. We reserve the last 16 days for an out-of-sample time series analysis.

Table 1: **Summary Statistics for Corn, Soybean, and Wheat Futures and CSOs**

Grain	Product	Mean	Std. Dev.	Skew.	Kurt.	Min.	Max.	Op. Int.	Vol.
Corn	Nearby	6.65	0.58	-0.02	0.64	4.36	8.14	308,023	80,602
	Deferred	5.86	0.50	0.35	-0.43	4.43	7.14	28,4347	41,479
	CSO	0.23	0.13	2.16	9.84	0.01	1.11	1,244	143
Soybeans	Nearby	13.45	0.96	-1.03	0.74	9.92	15.05	151,030	53,701
	Deferred	13.01	0.79	-1.34	2.06	9.65	13.99	151,276	28,567
	CSO	0.18	0.12	1.44	2.83	0.00	0.68	1,422	178
Wheat	Nearby	7.78	1.08	-0.15	-1.42	6.18	9.27	188,312	45,291
	Deferred	8.05	0.80	-0.04	-1.00	6.79	9.34	52,103	4,530
	CSO	0.14	0.10	0.92	0.44	0.01	0.43	61	38

We first notice that, corn and soybean futures prices exhibit backwardation whereas wheat futures prices experience contango during the sample period. This feature of the data is consistent with the dominance of CSOs with positive (negative) actual spread for corn and soybeans (wheat) as we shall see in Table 2. Furthermore, the volatility of the nearby contracts is slightly higher than in the deferred. This can be explained by the fact the nearby is much more heavily traded than the deferred and any new information will affect the nearby price much more than the deferred.

Table 2 reports the number of observations according to the type of options and the strike. For both corn and soybeans, positive strikes dominate trading while for wheat, negative strikes are more prevalent. The dominance of calls with positive strikes in the corn and soybeans is a direct result of the price of the nearby futures contract being consistently above the deferred contract. The opposite is true for the wheat market: the dominance of put options with negative strikes . Market participants need to hedge the risk of this consistently positive (negative) spread widening or narrowing. Also, in general, most trading activity in options occurs near-the-money, where strikes are close to the calendar spread itself. For the estimation purpose, we employ the most at-the-money options[5], which amounts to 268 calls for corn, 225 calls for soybeans and 85 puts for wheat.

Table 2: **Distribution of CSOs by Strike** K

Product	Type	K < 0	K = 0	K > 0	Total
Corn	Call	0	2	647	649
	Put	1	78	427	506
	#	1	80	1074	1155
Soybeans	Call	0	8	634	642
	Put	17	53	333	403
	#	17	61	967	1045
Wheat	Call	11	6	39	56
	Put	101	24	12	137
	#	112	30	51	193

3.2 Methodology for Parameter Estimation

Parameter estimation for the GBM model is straightforward. We follow the traditional approach based on the minimization of the sum of squared errors over the sample period (see Bakshi, Cao and Chen 1997). The dollar error is defined as the difference between the Ravindran model price and the near-the-money market price.[6]

Parameter estimation for the SV model is more challenging, which involves the estimation of both the eight hyper parameters $\Theta := \{\sigma_1, \sigma_2, \rho_{12}, \rho_{1v}, \rho_{2v}, \kappa, \sigma_v, \mu\}$ and the implied state variable v_t. We largely follow Pan's (2002) "implied state-generalized method of moments" (IS-GMM) method, originally designed for options pricing on a single cash stock (index). The unique advantage of this method is that it requires only a time series of the underlying asset and one option price per day, while still reliably inferring the implied state (stochastic volatility). This method ideally fits our current problem for which we are faced with 3-4 strike prices for each maturity. We adapt the IS-GMM method for the problem of multi-asset options on agricultural commodities.

[5]The raw data consists of both floor and settlement close prices. The floor close represents the market's last trade while the settlement is a calculated value by the CME Group used for "mark-to-market" purposes. We thank Dwight Sanders (session chair), Wade Brorsen, Scott Irwin, Aaron Smith and other NCCC-134 2013 Conference attendees for this suggestion.

[6]We use Ravindran's pricing formula, as opposed to the FFT method, because it is extremely fast and very accurate due to its closed-form solution.

The IS-GMM estimation is conducted in a two-step procedure. In the first step, we minimize the pricing error (the FFT-based model price minus the actual market price) by changing the daily implied state using the root-finding secant method, conditional on a prior of structural hyper parameters. This step provides a vector of implied state values for the second step. The second step in the parameter estimation procedure is to use the daily implied state values to estimate the seven parameters by fixing σ_1 to be "1" for identification purpose. Given the time series of the implied state, we use the GMM method with a weighting matrix using the Bartlett kernel to estimate the hyper parameters. The detailed procedure is outlined in Pan (2002).

A total of 13 moments are derived from the following characteristic function of the joint distribution of log futures prices and stochastic volatility:

$$\phi(u_1, u_2, u_3) = \exp\left[iu_1 f_1 + iu_2 f_2 + iu_3 v\right].$$

More specifically, the 13 moments used in the GMM procedure include the first four moments for each futures (total of 8) and the first two moments of volatility, one cross-moment between futures, two cross-moments between futures and volatility. Their derivations are as follows:

$$\begin{aligned}
M_j^k &= \frac{\partial^k \phi(u_1, u_2, u_3)}{\partial u_j^k} i^k |_{u_1=u2=0} \text{ with } j = 1, 2 \text{ and } k = 1, \ldots, 4 \\
M_{cross} &= \frac{\partial^2 \phi_{sv}(u_1, u_2, u_3)}{\partial u_1 \partial u_2} i^2 |_{u_1=u2=0} \\
M_v^k &= \frac{\partial^k \phi(u_1, u_2, u_3)}{\partial u_3^k} i^k |_{u_3=0} \text{ with } k = 1, 2 \\
M_{vcross} &= \frac{\partial^2 \phi(u_1, u_2)}{\partial u_3 \partial u_j} i^2 |_{u_j=u_3=0} \text{ with } j = 1, 2
\end{aligned}$$

Since the characteristic function ϕ is in closed-form, so are all 13 moments. The actual derivation is performed by the symbolic math toolbox in Matlab®.

4 Results

We present the parameter estimates from the minimization procedure for the GBM and SV models. We conclude the section with an analysis of the in-sample and out-of-sample pricing errors.

4.1 Parameter Estimates and Analysis

Tables 3 provides the parameter estimates and standard errors for the GBM model. The σ_i's measure the volatility for the two stochastic processes. We find that the volatility of the nearby futures price is higher than the deferred futures price. The finding is consistent with Table 1, the nearby futures contract had a higher volatility than the deferred futures contract. These two results show that the nearby price changes more dramatically than the deferred.

This characteristic is the well-known Samuelson Hypothesis first proposed in Samuelson (1965) and detailed in Kalev and Duong (2008). The hypothesis states that volatility increases as futures approach the expiration date. An explanation for this increase is that any news, positive or negative, will affect the front month's price more than the deferred months.

Our results for GBM show that there is an almost perfect positive correlation between the nearby and deferred futures. This is reasonable given that the sole differentiating factor is time to expiration. Any changes to the nearby month should also change the deferred in the same direction. Positive news to hit the market should push both futures up while negative information will force them both down so that, either way, they move in tandem. We find that our estimates are significant having quite low standard errors for all three parameters.

Table 3: **GBM Model Parameter Estimates**

Grain	σ_1	σ_2	ρ
Corn	0.511	0.409	1.000
	(0.0309)	(0.0352)	(0.0019)
Soy	1.089	1.086	0.999
	(0.0296)	(0.0331)	(0.0009)
Wheat	0.183	0.016	1.000
	(0.0437)	(0.0454)	(0.0056)

Table 4: **SV Model Parameter Estimates**

Grain	σ_2	ρ_{12}	ρ_{1v}	ρ_{2v}	κ	σ_v	μ
Corn	0.582	0.751	0.012	0.019	1.683	0.908	0.050
	(0.000)	(0.019)	(0.000)	(0.000)	(0.021)	(0.010)	(0.028)
Soybeans	0.795	0.788	0.035	0.035	1.947	1.458	0.019
	(0.000)	(0.003)	(0.000)	(0.000)	(0.000)	(0.001)	(0.003)
Wheat	0.794	0.733	0.156	0.154	2.277	1.128	0.012
	(0.000)	(0.004)	(0.000)	(0.000)	(0.000)	(0.001)	(0.002)

Figure 1 shows the estimated values of the implied volatility state variable for the SV model. It is clear that volatilities of all three futures show randomness over time, justifying the existence of stochastic volatility. Daily implied volatility averages and standard deviations are 0.0965 and (0.1289) for corn, 0.0296 and (0.0583) for soybeans, and 0.2852 and (0.0177) for wheat.

Table 4 provides the parameter estimates and standard errors for the SV model. First, note that all parameter estimates are statistically significant. For identification purposes, we set σ_1 equal to 1 and let σ_2 fluctuate. As with the GBM estimates, the SV estimates of

σ_1 are higher than σ_2 indicating backwardization in futures volatility and further evidence of the Samuelson Hypothesis. The futures price correlation parameter ρ is large indicating strong correlation between the futures prices, a similar result to the GBM estimate.

The correlation in the corn and soybeans markets between the futures price and volatility indicates a slight "inverse leverage effect." This effect results in the increase in prices as volatility increases as opposed to a decrease in prices as volatility increases which is readily observable in equities. An increase in agricultural market volatility is most likely a result in the decrease in the supply of a commodity. The decrease in supply triggers an increase in price. Because both parameters have the same positive sign, both the nearby and deferred futures in the corn and soybean markets move in tandem with volatility shocks.

The price-volatility parameters in the wheat market have different signs. The positive sign for the nearby indicates an increase in price as volatility increase while the negative sign in the deferred indicates the opposite reaction. During this time, the wheat market experienced a protracted period of contango. As a result, a volatility shock will force the nearby price higher and the deferred lower.

The speed of mean reversion in the volatility process is moderately slow. We use $\ln(2)/\kappa$ to measure the half-time of mean reversion which range from three to five months. The old crop-new crop CSO experiences consistent trading leading up to expiration after which trading is sparse for a period of time spanning approximately six months. This lack of trading prolongs the half-time so that any shock requires months for voltility mean reversion.

The volatility of volatility, σ_v, is significantly higher than the nearby and deferred volatility σ_i with the exception of the nearby corn future. The estimation of the long run mean of variance (μ) is consistent with the annualized volatility of the nearby and deferred futures. The estimates range from 11% to 22% annualized volatility and is commensurate with the annualized historical realized volatility which ranges from 26% to 41%.

4.2 Pricing Error Analysis

We compute the in-sample and out-of-sample relative pricing errors for the GBM and SV models. The in-sample errors are from the data set used in parameter estimation. The out-of-sample data set include any observations traded on the same day as the in-sample observations but not used in parameter estimation. The dollar error for the i^{th} observation is defined as

$$|\hat{P}_i - P_i|$$

with $\hat{P}_i$ the model option price and P_i the market option price. The percentage error is defined as the dollar error divided by P_i.

Figure 2 presents the dollar and percentage errors for the GBM model. In general, the GBM model produces high and erratic dollar and percentage errors for all three contracts.

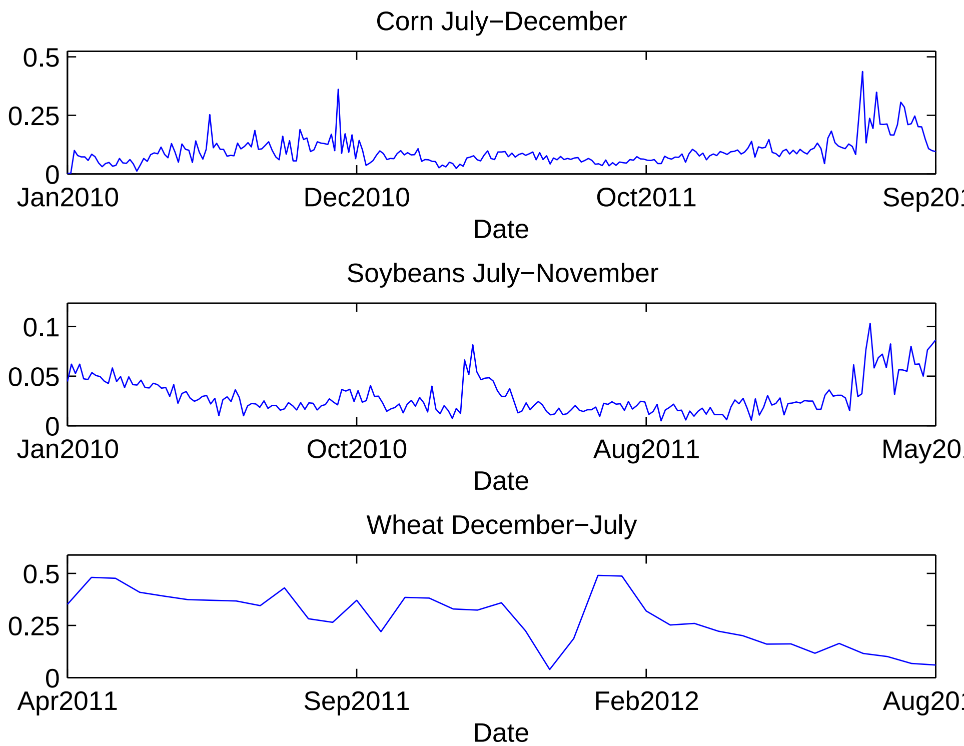

Figure 1: **Daily Volatility**

There are periods of relatively high errors indicating the inability of the GBM model to cope with changes in market volatility.

Figures 3 and 4 present the dollar and percentage errors for the SV model, respectively. On the left column of both figures, the SV errors with all observations included appear largely zero, with a few exceptions. For ease of interpretation and readability, these outliers for the SV errors have been removed for comparison (see the right column). Outliers in the dollar error graphs are any observations that are greater than 0.01 or 1 cent. This results in the removal of five observations from corn, six from soybeans, and one from wheat. Outliers in the percentage error graphs are any observations greater than 10^{-5} or 0.001 percent. This results in the removal of twelve observations from corn, eleven from soybeans, and five from wheat. The SV errors are, on average, dramatically lower than the GBM errors. The pricing performance for the in-sample data set is much better for SV than GBM given the consistently small pricing errors. This indicates that SV model copes better with changing market dynamics than GBM. Therefore, latent volatility is a necessary component of market dynamics of agricultural CSOs.

Tables 5 to 7 present the statistics for the in-sample errors for absolute and signed dollar and percentage errors for GBM and SV models. Note that, even though the pricing error outliers are included, the SV absolute dollar and percentage error means are considerably less than the GBM means and also have lower standard deviations. Specifically, the percentage errors for SV corn, soybeans and wheat are, respectively, 0.79%, 0.75% and 1.17%. These errors are, on average, 4% of the GBM errors. These results indicate the SV model is producing consistently lower absolute errors than the GBM model.

Table 5: **In-Sample Errors for Corn July-Dec**

Error Type	Mean	Std. Dev.	Min	Max
GBM $	-0.0187	0.0488	-0.2002	0.1022
SV $	0.0003	0.0108	-0.1198	0.0957
GBM %	-0.1219	0.3421	-0.9596	2.1184
SV %	0.0050	0.0608	-0.3603	0.5888
GBM Abs. $	0.0413	0.0319	0.0000	0.2002
SV Abs. $	0.0013	0.0108	0.0000	0.1198
GBM Abs. %	0.2497	0.2634	0.0000	2.1184
SV Abs. %	0.0079	0.0605	0.0000	0.5888

Note: At-the-money call options from January 7^{th}, 2010 to September 27^{th}, 2012, are used for estimation. "Dollar" errors are defined as the absolute difference between the theoretical model and empirically observed option prices. "Percentage" errors are defined as the Dollar errors divided by the observed options price.

Table 8 presents the out-of-sample GBM and SV dollar errors categorized according to signed moneyness. For wheat, the SV model produces better results for both puts and calls. For the corn and soybean markets, the SV calls produce consistently smaller errors than GBM calls with the exception of soybean deep out-of-the-money calls in which the GBM average error (1.4 cents) is similar to the SV (1.6 cents). Compared to the GBM model, the SV model generates smaller errors 67% of time and same errors 25% of time, and slightly larger errors 8% of time for corn CSO puts. The SV model fares better than the GBM model 27% of time, while the difference between the two models is about 0.4 cents. In general, the SV model is able to produce better out-of-sample dollar errors than the GBM model with few exceptions, for which the difference is statistically and economically negligible.

Figure 5 presents the time-series absolute dollar errors. The time series data set is the most at-the-money call for corn and soybeans and put for wheat for the last 16 days of the entire data set. In general, the SV errors are lower than the GBM errors indicating that the SV model is better at generating prices. The errors for both models are relative high compared with the cross-sectional out-of-sample errors. Volatility input for the SV model is the estimation for μ.

4.3 Hedging Error Analysis

We conduct simple delta hedge analysis from three perspectives: combined (comb), market makers (MM) and agribusiness buyers (buyer). The calculation of delta is based on

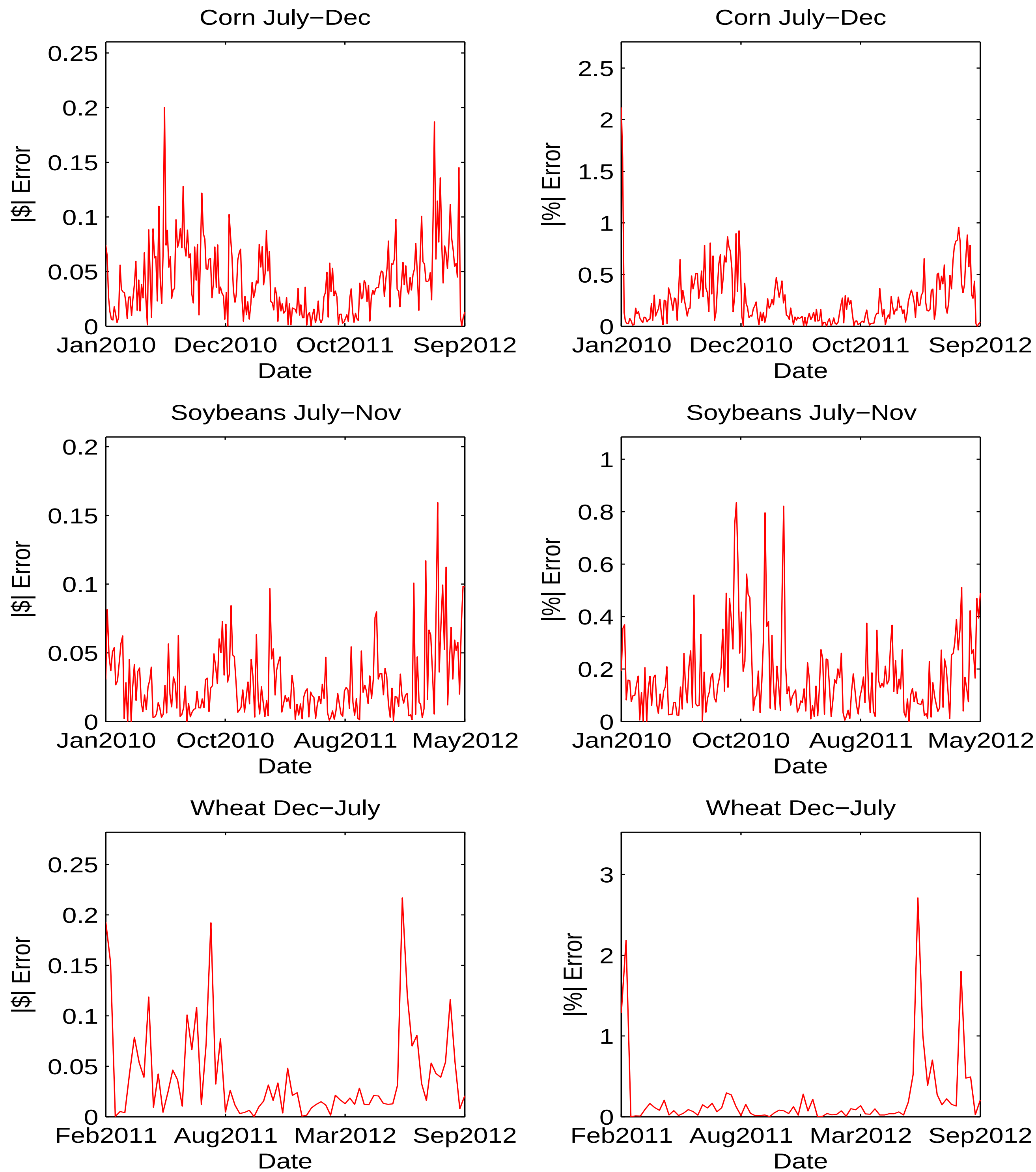

Figure 2: Absolute Dollar and Percentage Errors for the GBM Model
Note: In-sample absolute dollar and percent errors. "Dollar |$| errors on the left column are the absolute difference between the theoretical and observed options prices. "Percentage |%| errors on the right column are the dollar errors divided by the observed options prices.

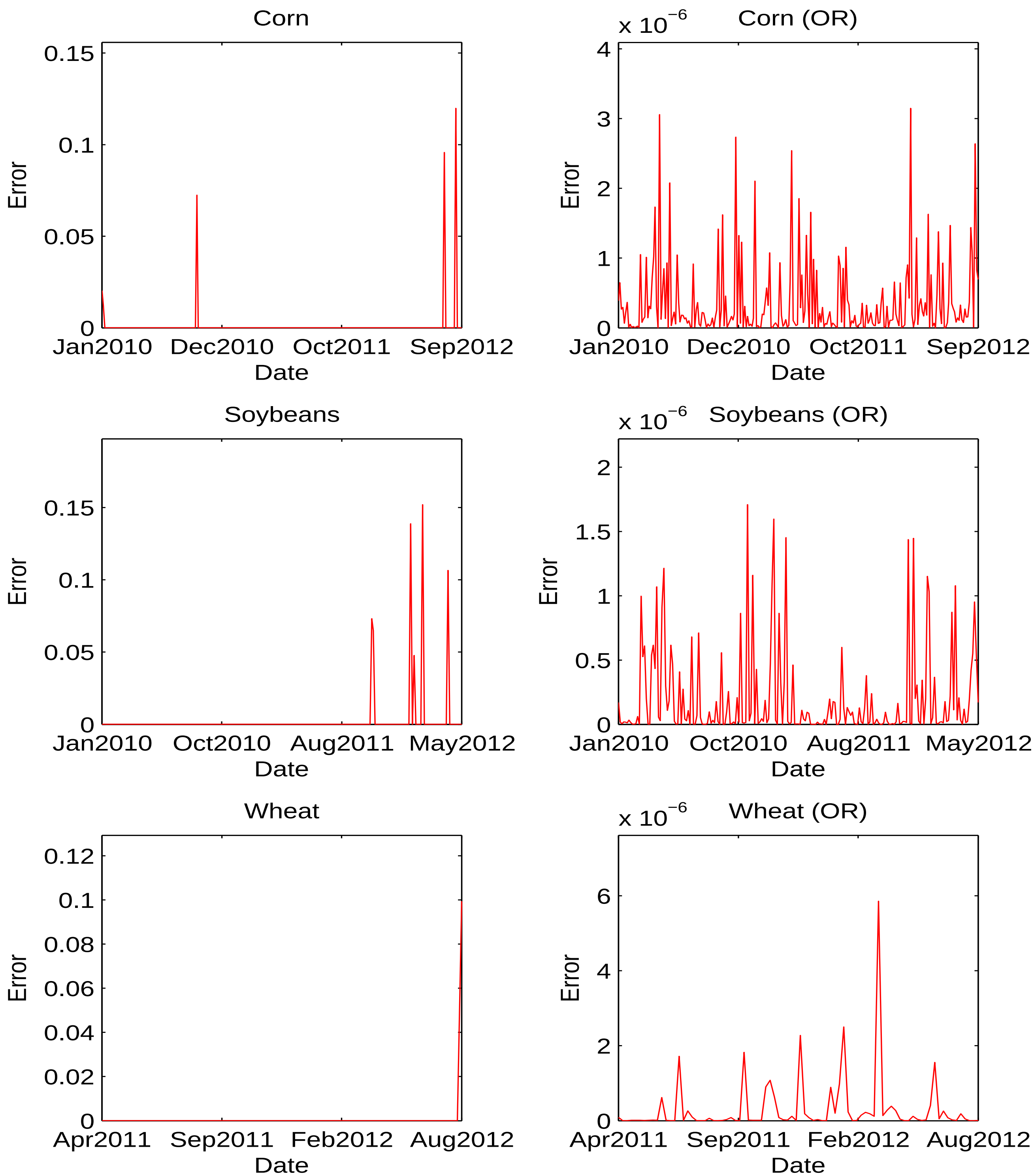

Figure 3: Absolute Dollar Errors for the SV Model
Note: In-sample absolute dollar errors. "Dollar errors are the absolute difference between the theoretical and observed options prices. In the left column, all observations are included. In the right column with "outliers removed" (OR), five, six and one outlier(s) with errors greater than 0.01 are removed for corn, soybeans and wheat, respectively, for better scaling.

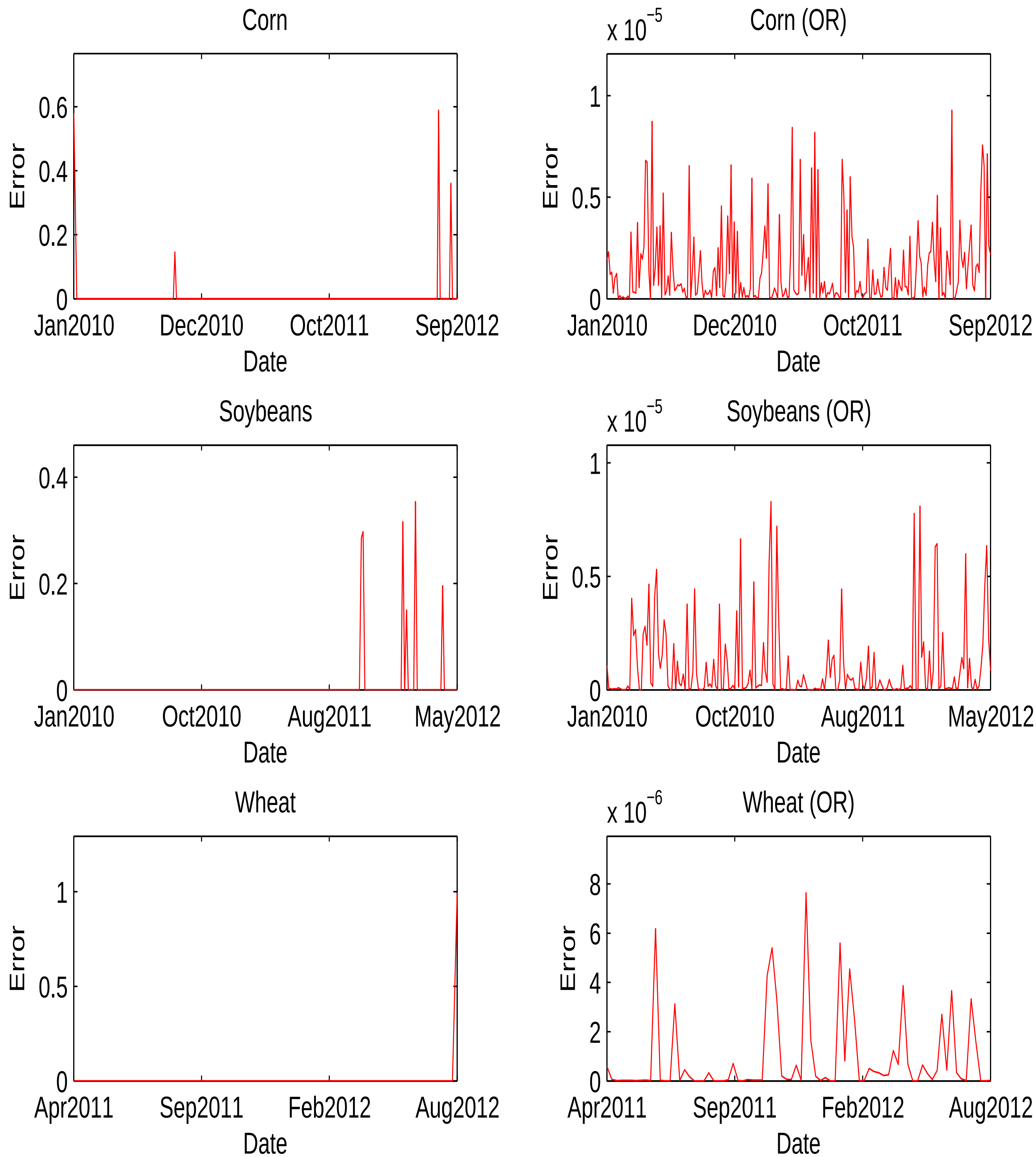

Figure 4: Absolute Percentage Errors for the SV Model

Note: In-sample absolute percentage errors. “Percentage errors are the dollar errors divided by the observed options prices. In the left column, all observations are included. In the right column with ”outliers removed” (OR), twelve, eleven and five outliers with errors greater than 10^{-5} are removed for corn, soybeans and wheat, respectively, for better scaling.

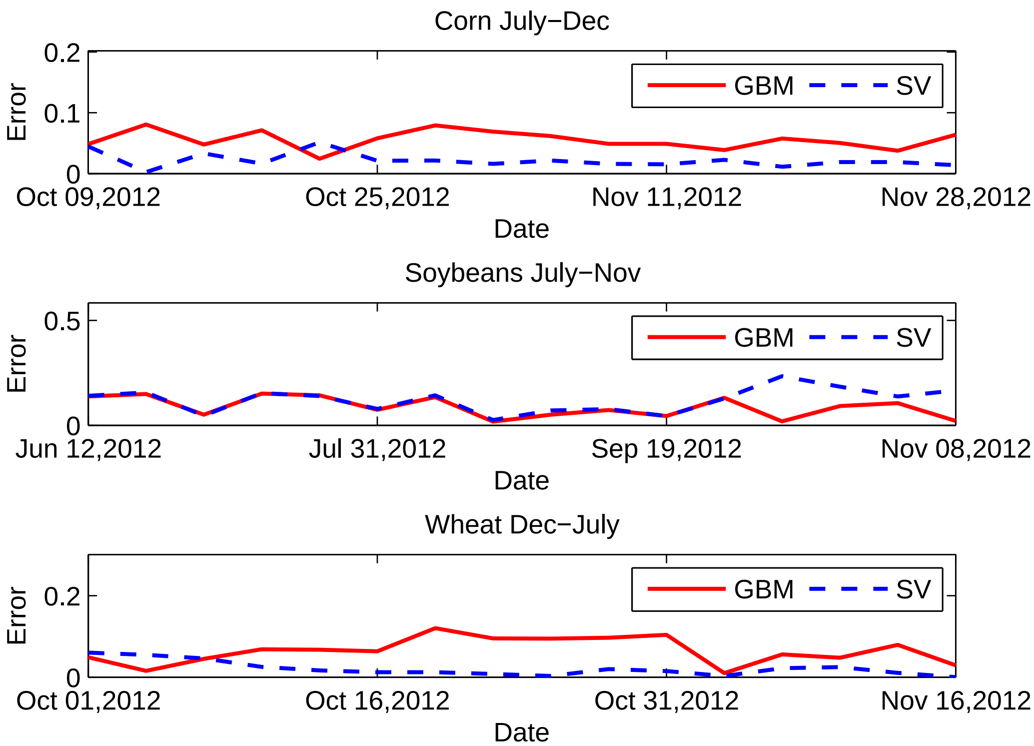

Figure 5: Out-of-Sample Time Series Dollar Pricing Errors

Note: The time series data set is composed of the last 16 days of the full data set. For corn and soybeans, the most at-the-money calls are used. For wheat, the most at-the-money puts are used. The volatility input for the SV model is 0.05 for corn, 0.019 for soybeans, and 0.012 for wheat. These values are the estimates fo rthe long-run volatility mean μ.

Table 6: **In-Sample Errors for Soy July-Nov**

Error Type	Mean	Std. Dev.	Min	Max
GBM $	-0.0033	0.0380	-0.1594	0.1170
SV $	0.0027	0.0174	-0.0000	0.1519
GBM %	-0.0391	0.2279	-0.8349	0.4824
SV %	0.0075	0.0458	-0.0000	0.3543
GBM Abs. $	0.0281	0.0258	0.0000	0.1594
SV Abs. $	0.0027	0.0174	0.0000	0.1519
GBM Abs. %	0.1709	0.1553	0.0002	0.8349
SV Abs. %	0.0075	0.0458	0.0000	0.3543

Note: At-the-money call options from January 8^{th}, 2010 to May 29^{th}, 2012, are used for estimation. "Dollar" errors are defined as the absolute difference between the theoretical model and observed option prices. "Percentage" errors are defined as the Dollar errors divided by the observed options price.

finite differences in CSO prices. The combined delta hedge takes the same number of nearby and deferred contracts with opposite long/short positions to hedge the CSO, defined as $H^C_{Comb} = Opt_{market} - \Delta(F_N - F_D)$, where Opt_{market}, F_N, F_D, Δ are the CSO option price, nearby futures price, deferred futures price and the delta hedge ratio. Market makers hedge out all delta risk using optimal, often different, numbers of nearby and deferred contracts. The "MM" delta-hedged portfolio value with one call CSO is defined as $H^C_{MM} = Opt_{market} - \Delta_N F_N + \Delta_D F_D$. This hedge gives the proportion of nearby and deferred futures needed to completely hedge the CSO. The efficacy of the MM hedge is given by the distance from 0. The closer to 0 the hedge outcome is, the better the market maker is able to create a complete delta hedge in which there is no market price risk.

Agribusinesses, on the other hand, need to hedge the rollover risk in the nearby contract, while maintaining the exposure to the deferred contract. The "buyer" call hedge is defined as $H^C_{buyer} = Opt_{market}/\Delta_N - F_N$. This hedge gives the number of CSOs required to roll over the nearby contract position into the deferred contract. Therefore, the efficacy of the buyer hedge, however, is measured by how close the hedge is to the deferred price.

Based on the empirically computed hedge ratios, we report the hedging results in Tables 9, 10, and 11. For all three products, the combined SV hedge is better than the GBM given that the values are all closer to 0 indicating that the SV model is better at hedging than the GBM model. The wheat MM SV is closer to 0 than the GBM. The soybean buyer hedge is closer to the average deferred futures price than the GBM. In sum, the SV model provides more reliable hedging than the GBM model.

Table 7: **In-Sample Errors for Wheat Dec-July**

Error Type	Mean	Std. Dev.	Min	Max
GBM $	-0.0001	0.0612	-0.1922	0.2169
SV $	-0.0012	0.0108	-0.0994	0.0000
GBM %	0.1265	0.5042	-0.5224	2.7111
SV %	-0.0117	0.1079	-0.9943	0.0012
GBM Abs. $	0.0395	0.0466	0.0001	0.2169
SV Abs. $	0.0012	0.0108	0.0000	0.0994
GBM Abs. %	0.2269	0.4671	0.0003	2.7111
SV Abs. %	0.0117	0.1078	0.0000	0.9943

Note: At-the-money call options from February 11^{th}, 2011 to September 24^{th}, 2012, are used for estimation. "Dollar $" errors are defined as the absolute difference between the theoretical model and empirically observed option prices. "Percentage %" errors are defined as the Dollar errors divided by the observed options price.

5 Conclusion

Agricultural CSOs help alleviate spread risk associated with rolling over futures contracts. Both consumers and suppliers of grains, such as ethanol plants and grain elevators, are interested in risk-managing the rollover risk through CSOs. Although there is theoretical research on related spread options on cash assets, there is lack of theoretical and empirical research aiming to understand the pricing and hedging of agricultural CSOs. We fill the gap by solving the CSO pricing problem under the geometric Brownian motion (GBM) and stochastic volatility (SV) models. We further employ the IS-GMM estimation, which has not been applied to multi-asset or futures data, to overcome the data limitation. We find the SV model can fit the market data extremely well, with less than 1% error on average for corn and soybeans, 1.2% error for wheat. Furthermore, the pricing and hedging performance of the SV model are superior to the benchmark GBM model, both in and out of sample, with only one exception where the out-of-sample hedging error for the GBM model for market makers is slightly better than the SV model. Our empirical results lend support for agribusinesses and the market makers to adopt the SV model for pricing and hedging grain CSOs. Another implication for the exchange (CME) is to employ the SV model, instead of the Black-Scholes-type GBM model for determining the settlement prices. The research can be extended in the following directions: (1) investigating the empirical performance of our models on more CSO contracts; (2) adding a jump term and other model structures to the SV model to see if the pricing and hedging performances are improved significantly.

Table 8: **Cross-Sectional Out-of-Sample Dollar Errors Across Strikes** (m)

			$m < -0.5$	$-0.5 \le m < 0$	$0 \le m < 0.5$	$m > 0.5$
Corn	Call	N	27	98	30	1
		GBM	0.029	0.031	0.031	0.074
		SV	0.018	0.017	0.016	0.014
	Put	N	15	122	45	0
		GBM	0.014	0.032	0.042	-
		SV	0.019	0.028	0.042	-
Soy	Call	N	2	38	73	49
		GBM	0.015	0.016	0.018	0.014
		SV	0.012	0.015	0.015	0.016
	Put	N	2	85	35	8
		GBM	0.024	0.027	0.033	0.031
		SV	0.033	0.030	0.029	0.033
Wheat	Call	N	0	0	16	21
		GBM	-	-	0.039	0.027
		SV	-	-	0.017	0.012
	Put	N	0	0	27	4
		GBM	-	-	0.031	0.017
		SV	-	-	0.028	0.023

Note: "Out-of-sample" observations are those options contracts which are not at-the-money during the in-sample time period. "N" denotes the number of options. "Dollar errors are defined as the absolute difference between the theoretical model and observed option prices.

Appendices

A Derivation of the Characteristic Function for the Stochastic Volatility Model

Our objective is to find

$$E_t^{\mathbb{Q}}[e^{iu_1 \log S_1(T_1) + iu_2 \log S_2(T_2)}] = E_t^{\mathbb{Q}}[e^{iu_1 s_1(T_1) + iu_2 s_2(T_2)}]$$

From our previous paper's derivation, let

$$\Psi(u_1, u_2, t, T_0, T_1, T_2) = E_{t_1}^{\mathbb{Q}}[e^{iu_1 s_1(T_1) + iu_2 s_2(T_2)}]$$

Since we are pricing an option on a calendar spread, the underlying futures have two different times to expirations: t_1 and t_2. The structure of the calendar option spread is such that expiration coincides with the nearby future. Therefore, we denote time to expiration for the option as t_1 which is also the time to expiration for the nearby future.

Table 9: **CSO Hedging Mean and Standard Deviation: Corn**

GBM	In-Sample	Out-of-Sample Calls	Out-of-Sample Puts
Comb	0.2054	0.0918	0.1449
	(0.0991)	(0.0630)	(0.0924)
MM	-0.2913	-0.1818	0.0028
	(0.1199)	(0.1250)	(0.0972)
Buyer	-6.1442	-5.6083	6.0632
	(0.5514)	(2.6332)	(0.7149)
SV	In-Sample	Out-of-Sample Calls	Out-of-Sample Puts
Comb	0.1957	0.0835	0.1353
	(0.0975)	(0.0612)	(0.0930)
MM	-0.3509	-0.2388	0.0712
	(0.0988)	(0.1120)	(0.0991)
Buyer	-6.1505	-6.1099	6.1418
	(0.5412)	(0.5499)	(0.5418)

Note: "Comb" is the combined delta hedge which is defined as $H^C_{Comb} = Opt_{market} - \Delta(F_N - F_D)$. "MM" is the market maker's delta hedge defined as $H^C_{MM} = Opt_{market} - \Delta_N F_N + \Delta_D F_D$. "Buyer" is the agribusiness buyer's delta hedge defined as $H^C_{buyer} = Opt_{market}/\Delta_N - F_N$. The combined and market maker hedges are better the closer to zero. Buyer hedges are better the closer to the average deferred futures price of 5.86. Standard deviations are in parentheses and are under the mean of the hedge.

Assume the solution is of the form

$$\Psi(u_1, u_2, t, T_0, T_1, T_2) = e^{A(T_0-t)+B(T_0-t)v(t)+\mathrm{i}u_1 s_1(T_1)+\mathrm{i}u_2 s_2(T_2)}$$

Let $\tau = T_0 - t$. We can find $d\Psi(t, v, s_1, s_2)$ with the following

$$\begin{aligned} d\Psi(t, v, s_1, s_2) &= \Psi_t dt + \Psi_v dv + \Psi_{s_1} ds_1 + \Psi_{s_2} ds_2 + \frac{1}{2}\Psi_{vv}(dv)^2 \\ &+\frac{1}{2}\Psi_{s_1 s_1}(ds_1)^2 + \frac{1}{2}\Psi_{s_2 s_2}(ds_2)^2 + \Psi_{vs_1} dv ds_1 \\ &+\Psi_{vs_2} dv ds_2 + \Psi_{s_1 s_2} ds_1 ds_2 \end{aligned}$$

Table 10: **CSO Hedging Mean and Standard Deviation: Soybeans**

GBM	In-Sample	Out-of-Sample Calls	Out-of-Sample Puts
Comb	0.1829	0.0685	0.1821
	(0.1019)	(0.0533)	(0.1667)
MM	-0.1662	-0.1092	0.0906
	(0.1394)	(0.0975)	(0.1862)
Buyer	-13.0942	-13.1765	13.2980
	(0.9044)	(1.0660)	(0.7693)
SV	In-Sample	Out-of-Sample Calls	Out-of-Sample Puts
Comb	0.1795	0.0661	0.1779
	(0.0989)	(0.0508)	(0.1664)
MM	-0.2076	-0.1054	0.1569
	(0.1749)	(0.0972)	(0.1638)
Buyer	-13.0048	-12.9545	13.2926
	(0.9268)	(0.8272)	(0.7825)

Note: "Comb" is the combined delta hedge which is defined as $H^C_{Comb} = Opt_{market} - \Delta(F_N - F_D)$. "MM" is the market maker's delta hedge defined as $H^C_{MM} = Opt_{market} - \Delta_N F_N + \Delta_D F_D$. "Buyer" is the agribusiness buyer's delta hedge defined as $H^C_{buyer} = Opt_{market}/\Delta_N - F_N$. The combined and market maker hedges are better the closer to zero. Buyer hedges are better the closer to the average deferred futures price of 13.01. Standard deviations are in parentheses and are under the mean of the hedge.

In order to solve for $E^{\mathbb{Q}}_t\left[\frac{d\Psi(t)}{\Psi(t)}\right]$ we need to find $E^{\mathbb{Q}}_t[dv]$, $E^{\mathbb{Q}}_t[ds_1]$, $E^{\mathbb{Q}}_t[ds_2]$, , $E^{\mathbb{Q}}_t\left[(dv)^2\right]$, $E^{\mathbb{Q}}_t\left[(ds_1)^2\right]$, $E^{\mathbb{Q}}_t\left[(ds_2)^2\right]$,$E^{\mathbb{Q}}_t[dvds_1]$, $E^{\mathbb{Q}}_t[dvds_2]$, $E^{\mathbb{Q}}_t[ds_1ds_2]$. Then we have:

$$
\begin{aligned}
E^{\mathbb{Q}}_t\left[\frac{d\Psi(t_1)}{\Psi(t_1)}\right] \quad = \quad & E^{\mathbb{Q}}_t\left[\left(\frac{-dA(\tau)}{d\tau} - \frac{dB(\tau)}{d\tau}v(t)\right)dt + B(\tau)dv\right. \\
& + (\mathrm{i}u_1)\,ds_1 + (\mathrm{i}u_2)\,ds_2 + (B(\tau))^2\,(dv)^2 \\
& + \frac{1}{2}\,(\mathrm{i}u_1)^2\,(ds_1)^2 - \frac{1}{2}\,(\mathrm{i}u_2)^2\,(ds_2)^2 \\
& + B(\tau)\,(\mathrm{i}u_1)\,dvds_1 - B(\tau)\,(\mathrm{i}u_2)\,dvds_2 \\
& \left. - \left(\mathrm{i}^2u_1u_2\right)ds_1ds_2\right]
\end{aligned}
$$

Table 11: **CSO Hedging Mean and Standard Deviation: Wheat**

GBM	In-Sample	Out-of-Sample Calls	Out-of-Sample Puts
Comb	0.3195	0.3260	0.1017
	(0.1989)	(0.1402)	(0.0740)
MM	-0.1345	-0.1637	0.2266
	(0.1372)	(0.1288)	(0.1857)
Buyer	-7.4987	-7.3307	7.5853
	(0.6613)	(0.7070)	(0.6551)
SV	In-Sample	Out-of-Sample Calls	Out-of-Sample Puts
Comb	0.3178	0.3241	0.1002
	(0.1983)	(0.1402)	(0.0752)
MM	-0.1296	-0.1522	0.2245
	(0.1482)	(0.1550)	(0.1729)
Buyer	-7.4885	-7.3074	7.5883
	(0.6498)	(0.7058)	(0.5994)

Note: "Comb" is the combined delta hedge which is defined as $H^C_{Comb} = Opt_{market} - \Delta(F_N - F_D)$. "MM" is the market maker's delta hedge defined as $H^C_{MM} = Opt_{market} - \Delta_N F_N + \Delta_D F_D$. "Buyer" is the agribusiness buyer's delta hedge defined as $H^C_{buyer} = Opt_{market}/\Delta_N - F_N$. The combined and market maker hedges are better the closer to zero. Buyer hedges are better the closer to the average deferred futures price of 8.05. Standard deviations are in parentheses and are under the mean of the hedge.

Replacing the expectation, we get:

$$
\begin{aligned}
E_t^{\mathbb{Q}}\left[\frac{d\Psi(t)}{\Psi(t)}\right] &= \left(\frac{-dA(\tau)}{d\tau} - \frac{dB(\tau)}{d\tau}v(t)\right)dt + B(\tau)(\kappa(\mu - v)dt) \\
&+ (\mathrm{i}u_1)\left((r_1 - \frac{1}{2}\sigma_1^2 v)dt\right) \\
&- (\mathrm{i}u_2)\left((r_2 - \frac{1}{2}\sigma_2^2 v)dt\right) \\
&+ (B(\tau))^2\left(\sigma_v^2 v dt\right) \\
&+ \frac{1}{2}(\mathrm{i}u_1)^2\left(\sigma_1^2 v dt\right) \\
&- \frac{1}{2}(\mathrm{i}u_2)^2\left(\sigma_2^2 v dt\right) \\
&+ B(\tau)(\mathrm{i}u_1)(\sigma_1\sigma_v v\rho_{13}dt) - B(\tau)(\mathrm{i}u_2)(\sigma_2\sigma_v v\rho_{23}dt) \\
&- \left(\mathrm{i}^2 u_1 u_2\right)\sigma_1\sigma_2 v\rho_{12}dt
\end{aligned}
$$

We now solve the following:

$$\frac{1}{dt}E_t^{\mathbb{Q}}\left[\frac{d\Psi(t)}{\Psi(t)}\right] = 0$$

We will collect all the terms associated with v and all of the constant terms and set each set equal to 0. Then we have

$$\begin{aligned}
0 &= \frac{-dA(\tau)}{d\tau} + B(\tau)\kappa\mu + (\mathrm{i}u_1)(r_1) + (\mathrm{i}u_2)(r_2) \\
0 &= -\frac{dB(\tau)}{d\tau}v + (B(\tau))^2(\sigma_v^2 v) - B(\tau)\kappa v + B(\tau)(\mathrm{i}u_1)(\sigma_1\sigma_v v\rho_{1,v}) + B(\tau)(\mathrm{i}u_2)(\sigma_2\sigma_v v\rho_{2,v}) \\
&\quad -\mathrm{i}u_1\frac{1}{2}\sigma_1^2 v - \mathrm{i}u_2\frac{1}{2}\sigma_2^2 v + \frac{1}{2}(\mathrm{i}u_1)^2(\sigma_1^2 v) \\
&\quad +\frac{1}{2}(\mathrm{i}u_2)^2(\sigma_2^2 v) + \left(\mathrm{i}^2 u_1 u_2\right)\sigma_1\sigma_2 v\rho_{1,2}
\end{aligned}$$

Dividing by v, we have

$$\begin{aligned}
\frac{dB(\tau)}{d\tau} &= B(\tau)^2\sigma_v^2 - B(\tau)(\kappa - \mathrm{i}(\rho_{13}\sigma_1 u_1 + \rho_{2,v}\sigma_2 u_2)\sigma_3) \\
&\quad -\frac{1}{2}\left[\sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 + \mathrm{i}(\sigma_1^2 u_1 + \sigma_2^2 u_2)\right] + \\
&\quad -(\rho_{12}\sigma_1\sigma_2 u_1 u_2)
\end{aligned}$$

B Derivation of $B(\tau)$

Now we have a Ricatti equation in the form of:

$$\frac{dB(\tau)}{d\tau} = \alpha B(\tau)^2 - \gamma B(\tau) + \zeta \tag{15}$$

with

$$\begin{aligned}
\alpha &= \sigma_v^2 \\
\gamma &= \kappa - \mathrm{i}(\rho_{13}\sigma_1 u_1)\sigma_v + \rho_{23}\sigma_2 u_2 \\
\zeta &= -\frac{1}{2}\left[\sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 + \mathrm{i}(\sigma_1^2 u_1 + \sigma_2^2 u_2)\right] \\
&\quad -\rho_{12}\sigma_1\sigma_2 u_1 u_2
\end{aligned}$$

Now let

$$y(t) = B(\tau)$$

and

$$\frac{dy(t)}{dt} = \frac{dB(\tau)}{d\tau}$$

This becomes

$$\frac{dB(\tau)}{\alpha B(\tau)^2 - \gamma B(\tau) + \zeta} = d\tau$$

Now we can use the following notation and rewrite as

$$\alpha y^2 - \gamma y + \zeta = \alpha(y - y_1)(y - y_2)$$

Now

$$\frac{dy}{(y-y_1)(y-y_2)} = \alpha dt$$

Implement partial fraction decomposition to get

$$\frac{1}{y-y_1} - \frac{1}{y-y_2} = \frac{y_1-y_2}{(y-y_1)(y-y_2)}$$

and

$$\frac{dy}{y-y_1} - \frac{dy}{y-y_2} = \alpha(y_1-y_2)dt$$

Now integrate

$$\int_{y_0}^{y} \left(\frac{ds}{s-y_1} - \frac{ds}{s-y_2}\right) = \int_{y_0}^{y} \alpha(y_1-y_2)dt$$

which becomes

$$\ln|y-y_1| - \ln|y_0-y_1| - \ln|y-y_2| + \ln|y_0-y_2| = \alpha(y_1-y_2)(y-y_0)$$

By rearranging we get

$$\ln\left|\frac{(y-y_1)(y_0-y_2)}{(y-y_2)(y_0-y_1)}\right| = \alpha(y_1-y_2)(y-y_0)$$

After exponentiating both sides we get

$$\frac{y-y_1}{y-y_2} = \left(\frac{y_0-y_1}{y_0-y_2}\right) e^{\alpha(y_1-y_2)(y-y_0)}.$$

We find that

$$\begin{aligned} y_1 &= \frac{\gamma-\theta}{2\alpha} \\ y_2 &= \frac{\gamma+\theta}{2\alpha}. \end{aligned}$$

Then

$$\alpha\left(\frac{\gamma-\theta}{2\alpha} + \frac{\gamma+\theta}{2\alpha}\right) = \gamma$$

and

$$\alpha\left(\frac{\gamma-\theta}{2\alpha}\right)\left(\frac{\gamma+\theta}{2\alpha}\right) = \zeta$$

From this we get

$$\frac{\gamma^2 - \theta^2}{4\alpha} = \zeta$$

which means

$$\theta = \sqrt{\gamma^2 - 4\alpha\zeta}.$$

Now

$$\begin{aligned} \alpha(y_1 - y_2)t &= \alpha\left(\frac{\gamma - \theta}{2\alpha} - \frac{\gamma + \theta}{2\alpha}\right)t \\ &= -\theta t \end{aligned}$$

and

$$\begin{aligned} y &= \frac{y_1 - y_1 e^{-\theta t}}{1 - \frac{y_1}{y_2} e^{-\theta t}} \\ &= \frac{2\zeta(1 - e^{-\theta t})}{2\theta + (\gamma - \theta)(1 - e^{-\theta t})} \end{aligned}$$

and we get

$$B(\tau) = \frac{2\zeta(1 - e^{-\theta T})}{2\theta - (\theta - \gamma)(1 - e^{-\theta T})}$$

C Derivation of $A(\tau)$

Now we have a

$$0 = \frac{-dA(\tau)}{d\tau} + B(\tau)\kappa\mu + (\mathrm{i}u_1)(r_1) + (\mathrm{i}u_2)(r_2)$$

with

$$B(\tau) = \frac{2\zeta(1 - e^{-\theta\tau})}{2\theta - (\theta - \gamma)(1 - e^{-\theta\tau})}$$

Then we solve in the following way:

$$\frac{dA(\tau)}{d\tau} = B(\tau)\kappa\mu + (\mathrm{i}u_1)(r_1) + (\mathrm{i}u_2)(r_2)$$

and

$$\begin{aligned} \int_0^{T_0} \frac{dA(\tau)}{d\tau} &= \int_0^{T_0} \Big[B(\tau)\kappa\mu + (\mathrm{i}u_1)(r_1) + (\mathrm{i}u_2)(r_2)\Big] d\tau \\ &= \kappa\mu \int_0^{T_0} B(\tau) d\tau + (\mathrm{i}u_1)(r_1)T + (\mathrm{i}u_2)(r_2)T \end{aligned}$$

Now

$$\int_0^{T_0} B(\tau)d\tau \quad = \quad \kappa\mu \int_0^{T_0} \frac{2\zeta(1-e^{-\theta\tau})}{2\theta-(\theta-\gamma)(1-e^{-\theta\tau})}d\tau$$

Let

$$\begin{aligned} u &= 1-e^{-\theta\tau} \\ du &= \theta e^{-\theta\tau} d\tau \end{aligned}$$

but

$$e^{-\theta\tau} = 1-u$$

so we have

$$\frac{du}{\theta(1-u)} = d\tau$$

Now, when we replace, we have:

$$\int_0^{T_0} B(\tau)d\tau = 2\zeta\kappa\mu \int \frac{u}{(2\theta-(\theta-\gamma)u)} \frac{du}{\theta(1-u)}$$

Rewrite the integral as

$$\int_0^{T_0} B(\tau)d\tau = \frac{2\zeta\kappa\mu}{\theta(\theta-\gamma)} \int \frac{u}{(u-\frac{2\theta}{\theta-\gamma})(u-1)}du$$

Using Vieta's formula we find

$$\frac{C}{u-\frac{2\theta}{\theta-\gamma}} + \frac{D}{u-1} = \frac{u}{(u-\frac{2\theta}{\theta-\gamma})(u-1)}$$

$$\begin{aligned} D &= \frac{1}{1-\frac{2\theta}{\theta-\gamma}} \\ &= \frac{-1}{(\theta+\gamma)(\theta-\gamma)} \end{aligned}$$

and

$$C = \frac{2\theta}{\theta+\gamma}$$

Now we have

$$
\begin{aligned}
\frac{2\zeta\kappa\mu}{\theta(\theta-\gamma)}\int_0^{T_0}\frac{u}{(u-\frac{2\theta}{\theta-\gamma})(u-1)}du &= \frac{2\zeta\kappa\mu}{\theta(\theta-\gamma)}\int_0^{T_0}\frac{\frac{2\theta}{\theta+\gamma}}{u-\frac{2\theta}{\theta-\gamma}}+\frac{\frac{\theta-\gamma}{-\gamma-\theta}}{u-1}du \\
&= \frac{2\zeta\kappa\mu}{\theta(\theta-\gamma)}\left[\frac{2\theta}{\theta+\gamma}\log\left(u-\frac{2\theta}{\theta-\gamma}\right)\Big|_0^{T_0}-\frac{\theta-\gamma}{\theta+\gamma}\log(u-1)\Big|_0^{T_0}\right]
\end{aligned}
$$

$$
= -\frac{\kappa\mu}{2\alpha}\left[2\log\left(\frac{2\theta-(\theta-\gamma)(1-e^{-\theta T})}{2\theta}\right)+(\theta-\gamma)T\right]
$$

Lastly, gathering the components back yields the characteristic function

$$
\begin{aligned}
\Phi(u;T,v_0) &= \exp\left(\left(\frac{2\zeta(1-e^{-\theta T})}{2\theta-(\theta-\gamma)(1-e^{-\theta T})}\right)v_0+(\mathrm{i}u_1)(r_1)T+(\mathrm{i}u_2)(r_2)T\right. \\
&\quad \left.-\frac{\kappa\mu}{2\alpha}\left[2\log\left(\frac{2\theta-(\theta-\gamma)(1-e^{-\theta T})}{2\theta}\right)+(\theta-\gamma)T\right]\right).
\end{aligned}
$$

References

Gurdip Bakshi, Charles Cao, and Zhiwu Chen. Empirical performance of alternative option pricing models. *The Journal of Finance*, 52(5):2003–2049, 1997.

R. Carmona and V. Durrleman. Pricing and Hedging Spread Options. *SIAM REVIEW*, 45: 627–685, 2003.

P. Carr and D. B. Madan. Option Valuation Using the Fast Fourier Transform. *The Journal of Computational Finance*, 2:61–73, 1999.

M.A.H. Dempster and S.S.G. Hong. Spread Option Valuation and the Fast fourier Transform. *Bachelier Congress 2000*, pages 203–220, 2002.

Fang Fang. Pricing Bermudan and American Options Using the FFT Method. *Master Thesis for Degree in Computational Engineering of Uni. Erlangen-Nuernberg, Germany*, 2006.

Héylette Geman and Vu-Nhat Nguyen. Soybean Inventory and Forward Curve Dynamics. *Management Science*, 51(7):1076–1091, July 2005.

David Heath, Robert Jarrow, and Andrew Morton. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60(1):77–105, 1992.

Steven L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):pp. 327–343, 1993. ISSN 08939454. URL `http://www.jstor.org/stable/2962057`.

T.R. Hurd and Z. Zhou. A Fourier Transform Method for Spread Option Pricing. *SIAM J. Financial Math*, 2013.

M. Prokopczuk J. Back and M. Rudolf. Seasonality and the Valuation of Commodity Options. *Journal of Banking and Finance*, 37(2):273–290, 2013.

P. S. Kalev and H. N. Duong. A test of the Samuelson hypothesis using realized range. *Journal of Futures Markets*, 28:680–696, 2008.

E. Kirk and J. Aron. Correlation in the Energy Markets. *Managing Energy Price Risk*, 1995.

C.C.W. Leentvaar and C. W. Oosterlee. Multi-Asset Option Pricing Using a Parallel Fourier-Based Technique. *Journal of Computational Finance*, 12(1):1–26, 2008.

Jun Pan. The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study. *Journal of Financial Economics*, 63:3–50, 2002.

F. Bervoets R. Lord, F. Fang and C. W. Oosterlee. A fast and accurate fft-based method for pricing early-exercise options under levy processes. *SIAM Journal Scientific Computation*, 30:1678–1705, 2008.

K. Ravindran. Low-Fat Spreads. *RISK*, 6:56–57, 1993.

P.A. Samuelson. Proof that Properly Anticipated Prices Fluctuate Randomly. *Industrial Management Review*, 6:41–49, 1965.

A. Schmitz, Z. Wang, and J. h. Kimn. A Jump Diffusion Model for Agricultural Commodities with Bayesian Analysis. *Journal of Futures Markets*, doi: 10.1002/fut.21597, 2013.

Eduardo Schwartz. The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging. *The Journal of Finance*, LII(3):923–973, July 1997.

Anders B. Trolle and Eduardo S. Schwartz. Unspanned Stochastic Volatility and the Pricing of Commodity Derivatives. *The Review of Financial Studies*, 22(11):4423–4461, 2009.