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Identifying Jumps and Systematic Risk in Futures

A variety of multivariate jump-diffusion models have been suggested as models of asset prices. This paper extends the literature on (joint) mixed jump-diffusion processes in futures markets by using the CRB index futures to represent systematic risk in commodity prices. We derive (joint) mixed bivariate normal distributions and likelihood functions for estimating the parameters of jump-diffusion processes. Likelihood ratio tests are used to select among nested models. The empirical results show the presence of downside jumps and significant systematic risk in wheat futures returns. Amin and Ng's (1993) model with a single counter of jumps fits better than other jump-diffusion processes considered. The jump components did not have significantly more systematic risk than the continuous component. In terms of wheat prices, one standard deviation jumps are 14 cents per bushel and two standard deviation jumps are 29 cents per bushel and are within the price limits. These jumps occur once in every six business days and are mostly crashes.
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Introduction

Previous studies show that futures returns have occasional large movements that result in asymmetric and leptokurtic distributions (Hudson, Leuthold and Sarassoro 1987; Hall, Brorsen, and Irwin 1989; Koekebakker and Gudbrand 2004). In this paper futures returns is defined as the percentage change in value from the closing price on one trading day to the closing price on the next trading day of a single contract. Since the quantity of a contract is fixed, the percentage change in price is equal to the percentage change in value. Discontinuous jumps in asset prices and time-varying volatility models are the two main approaches used to model extraordinary discrete price movements (Eraker 2004). In Merton's (1976) jump diffusion (JD) process, price changes occur at discrete points in time. The two weaknesses of the JD process that have been widely discussed in the literature are: (1) the model has only a single counter[1] of jumps and (2) the jump component represents only non-systematic risk and therefore a maintained hypothesis of the model is that all jump risk can be diversified. Empirical studies show high correlation between individual stock price volatility and market volatility (Jarrow and Rosenfeld 1984; Jorion 1988). If jumps in an individual asset are correlated with jumps in the overall market, then contrary to the maintained hypothesis of Merton's model, jump risk is systematic and could not be diversified. Amin and Ng (1993) derived an option pricing formula to account for stock return volatility that is both systematic and stochastic. In this model, the number of jumps in the consumption and asset price process are identical and are allowed to be correlated.

Camara's (2009) theoretical "two counters of jumps" model is a joint JD process of aggregate consumption and stock price. In this model, the jumps are separated into upside (bubbles) and downside (crashes) jumps with different intensity and distributional characteristics. Theoretically, Camara (2009) was able generalize Merton (1976) and Amin and Ng (1993) using "two counters of jumps" model. In this extended model, Camara (2009) included additional jump parameters in stock price and aggregate consumption. Camara's model, however has a potential estimation problem as it includes sixteen jump parameters along with other parameters that represent continuous price movements. As per Kou (2002) having so many

[1] Mathematically a counter process defines the number of arrivals that have occurred in the interval (0, t).

parameters in the model makes calibration difficult. In addition, if the jump magnitudes are small, the separation of jumps from continuous co-movements and estimation of parameters becomes less precise (Todorov and Bollerslev 2010). To circumvent this empirical problem, this paper includes four JD processes with fewer parameters that are nested in Camara's (2009) two counters of jumps model. The criteria[2] that are used in the selection of models are: (1) the model should be able explain the asymmetric and leptokurtic nature of returns, (2) the model should have an economic interpretation and practical implications, and (3) the likelihood function of the distribution should be mathematically tractable to compute the parameter estimates.

In Merton's (1976) model, jumps are firm specific and are not correlated with the stocks in general (*i.e.,* with the market). It is clear from the 1987 stock market crash that extreme events can influence all asset prices and market events. Similarly, commodities and commodity futures are systematically related to macroeconomic measures and the risk associated with jumps cannot be diversified (Hilliard and Reis 1999). The nature of the risk associated with jumps is not explicitly stated in Camara's (2009) model. Restricting the two counters of jumps model to have no jumps in consumption and single counter of jumps in stock price results in the JD economy of Merton (1976) with non-systematic jumps. The JD process of Amin and Ng (1993) with systematic jumps can be obtained by assuming a single counter of jumps in consumption and stock price. In the capital asset pricing model (CAPM) only systematic risk is rewarded. The current paper includes a joint JD process that also represents the extended one factor model of Todorov and Bollerslev (2010) with returns associated with continuous and discontinuous price moves. The model also allows to test whether the betas associated with these price moves are same. This more recently described JD process that includes both systematic and non-systematic jumps was published subsequent to Camara's (2009) paper.

The inelastic demand and dependence on weather increases the chance of extraordinary price movements in agricultural commodities. Some discrete incidence of large price changes is confined to a single commodity. For example, freeze damage in wheat may cause an extraordinary price change in wheat yet have little influence on other commodity prices. These firm (commodity) specific events result in non-systematic jumps in the returns. On the other hand, influences of macroeconomic variables like exchange rate fluctuations affect prices of all commodities and are systematic. In addition, Todorov and Bollerslev (2010) argued that the precision of beta estimates increases with less incidence of non-systematic risk. The current paper extends Amin and Ng (1993) by adding an uncorrelated jump to futures prices. We extend the futures literature by specifying a joint JD process with a single counter of jumps in market prices and two counters of jumps in futures prices. For the futures JD process, the jumps are defined with separate Poisson processes to represent systematic and non-systematic jumps. The model supports Todorov and Bollerslev's (2010) theoretical framework that extends the generic CAPM model to have two separate betas to represent the systematic risk attributable to continuous and discontinuous price moves.

The empirical objective of the paper is to select the most suitable JD process to model wheat futures prices and to estimate the systematic risk associated with the continuous and discontinuous components. In the current paper, we consider alternative stochastic processes that are nested in Camara's (2009) two counters of jumps model and estimate the parameters of the distributions. We contribute to the existing literature on futures by proposing a mixed bivariate normal distribution for simultaneously analyzing the price series. Both the continuous

[2] For detailed description of criteria, see Kou (2002).

movements and the discrete movements in each price series are modeled using normal and Poisson process respectively. This joint JD process includes correlated and uncorrelated jumps and models the interaction between wheat futures returns and CRB index returns. In addition, the paper also estimates the parameters of mixed univariate normal distributions with single and two counters of jumps in futures prices.

Jump diffusion processes that model systematic risk have several practical applications in portfolio and credit risk management. As the jumps are correlated with large number of assets, the presence of systematic risk reduces the gains from diversification and substantially increases the probability to lose while holding highly levered positions (Das and Uppal 2004). According to Duffie and Pan (2001), systematic movements also influence credit risk as happened during the recent financial crisis, especially with credit default swaps. Duffie and Pan (2001) used a jump-conditional value at risk (VaR) weighted by the probability of a given number of jumps for the analytical approximation of VaR. Jumps across the assets are systematic even in the commodity markets that makes it extremely difficult for grain trading firms to hedge the risk during large correlated price movements.

Maximum likelihood is used to estimate the parameters of the diffusion process using the Kansas City Board of Trade (KCBT) wheat futures prices and using the Commodity Research Bureau (CRB) index of futures prices to represent market returns. The Amin and Ng (1993) and Camara (2009) models are extensions of the consumption-based representative agent framework. In empirical analysis market prices are used instead of consumption growth (Amin and Ng, 2003). In this paper the CRB index of futures prices represents[3] the 'commodity market'. The empirical results show the presence of downside jumps and systematic jump risk in wheat futures prices. Amin and Ng's (1993) model fits the data better than other JD processes considered. The differences in beta estimates of continuous and discontinuous price moves were not statistically significant and failed to support Todorov and Bollerslev's (2010) extended CAPM framework.

Theoretical Model

This section introduces JD processes followed by more general univariate and bivariate models. All JD processes included in this section are restricted models of Camara's (2009) two counters of jump process. To begin, let the asset price (S_T) be a stochastic process with no discrete jumps so it can be represented as

$$(1)\quad S_T = \exp\left(\ln(S_0) + \eta_s T - \frac{\sigma_s^2}{2}T + \sigma_s B_s(T)\right).$$

where S_0 is the current futures price, $B_s(T) \sim N(0,T)$ is standard Brownian motion, η_s is the instantaneous expected price without any jumps, and σ_s^2 is the instantaneous variance of price without any jumps. The asset price is assumed to be non-negative and the T period return (logarithmic price relative) $r_T = \ln(S_T/S_0)$ is normally distributed as $r_T \sim N(\mu_f, \sigma_f^2)$ where $\mu_f = \eta_f - 0.5\sigma_f^2$ is the mean (drift) and $\sigma_f^2 = T\sigma_s^2$ is the variance.

[3] Note that an agribusiness firm wanting to use this approach in their VaR models would presumably want to create their own index that matched their own portfolio.

Univariate Mixed JD Process with Single Counter of Jumps

The number of extraordinary price changes follows a Poisson counting process $M(T)$ with mean arrival rate (intensity) λ and jump size $Y_{s,i}$. The JD process that includes both continuous and discontinuous changes in prices is

$$(2) \quad S_T = \exp\left(\ln(S_0) + \eta_s T - \frac{\sigma_s^2}{2} T + \sigma_s B_s(T) + \sum_{i=1}^{M(T)} Y_{s,i}\right).$$

In equation (2), the Brownian motion is a continuous process and the Poisson process is discontinuous. It is assumed that $B_s(T)$ and $Y_{s,i}$ are independent. The model reduces to geometric Brownian motion when there are no jumps. As per Jorion (1988), the probability density function of a mixed univariate normal distribution and a single counter of jumps is

$$(3) \quad f_r(r) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \frac{1}{\sqrt{2\pi(\sigma_f^2 + i\gamma_f^2)}} \exp\left[-\frac{(r_t - \mu_f - i\alpha_f)^2}{2(\sigma_f^2 + i\gamma_f^2)}\right].$$

Univariate Mixed JD Process with Two Counters of Jumps

Camara (2009) postulated a bivariate model with two separate Poisson distributed events to represent upside jumps and downside jumps. A univariate version of this richer, potentially more realistic and less restricted jump-diffusion process with two counters of jumps is

$$(4) \quad S_T = \exp\left(\ln(S_0) + \eta_s T - \frac{\sigma_s^2}{2} T + \sigma_s B_s(T) + \sum_{i=1}^{M(T)} Y_{su,i} + \sum_{j=1}^{N(T)} Y_{sd,j}\right).$$

where $Y_{su,i}$ and $Y_{sd,j}$ represent the upside jump and downside jumps, respectively, $M(T)$ is the Poisson counter process for upside jumps with intensity λ, and $N(T)$ is the Poisson counter process of downside jumps with intensity δ. In the price (logarithmic price relative) series, both jumps are normally distributed and can have different distributions each with different mean and variance. These price jumps can be represented as $Y_{su,i} \sim N(\alpha_{fu}, \gamma_{fu}^2)$ and $Y_{sd,i} \sim N(\alpha_{fd}, \gamma_{fd}^2)$ where, α_{fu} and α_{fd} are the means and γ_{fu}^2 and $\gamma_{fd}^2$are the variances of upside and downside jumps respectively.

For empirical estimation, the mean of upside jumps is restricted to be positive and the mean of downside jumps is restricted to be negative. In this model, the jump magnitudes are correlated if $cov(Y_{su,i}, Y_{sd,j}) \neq 0$ when $i = j$ and all other random variables are independent. The mixed bivariate normal pdf of returns with two counters of jumps is

$$(5) \quad f_r(r) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \sum_{j=0}^{\infty} \frac{e^{-\delta}\delta^j}{j!} \frac{1}{\sqrt{2\pi(\sigma_f^2 + i\gamma_{fu}^2 + j\gamma_{fd}^2 + 2\min(i,j)\,\upsilon_f)}}$$

$$\exp\left[-\frac{\left(r_t - \mu_f - (i\alpha_{fu} + j\alpha_{fd})\right)^2}{2(\sigma_f^2 + i\gamma_{fu}^2 + j\gamma_{fd}^2 + 2\min(i,j)\,\upsilon_f)}\right]$$

.

where υ_f represents the covariance of upside and downside jumps in returns. The covariance part in this mixed distribution merits more explanation. We assume that the continuous diffusion component (Brownian motion) is independent of all jumps[4]. The variance term of the mixed jump diffusion process is

$$(5a)\quad \mathrm{var}\left(r_T + \sum_{i=1}^{\infty} Y_{su,i} + \sum_{j=1}^{\infty} Y_{sd,j}\right) =$$

$$\mathrm{var}(r) + \mathrm{var}\left(\sum_{i=1}^{\infty} Y_{su,i}\right) + \mathrm{var}\left(\sum_{j=1}^{\infty} Y_{sd,j}\right) + 2\,\min\,(i,j)\mathrm{cov}\left(\sum_{i=1}^{\infty} Y_{su,i}\,, \sum_{j=1}^{\infty} Y_{sd,j}\right)$$

$$(5b)\quad \mathrm{var}\left(r_T + \sum_{i=1}^{\infty} Y_{su,i} + \sum_{j=1}^{\infty} Y_{sd,j}\right) = (\sigma_f^2 + i\gamma_{fu}^2 + j\gamma_{fd}^2 + 2\,\min(i,j)\,\upsilon_f).$$

As per the additive rule of covariance $\mathrm{cov}(X+Y,Z) = \mathrm{cov}(X,Z) + \mathrm{cov}(Y,Z)$ and by induction it can be stated as $\mathrm{cov}\left[\sum_{i=1}^{\infty} Y_{su,i}\,, \sum_{j=1}^{\infty} Y_{sd,j}\right] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{cov}\left[Y_{su,i}, Y_{sd,j}\right]$.

The minimum operator is to ensure equal number $(i = j)$ of jump magnitudes to carry out the estimation of covariance[5]. A JD process with separate upside and downside jumps is not a new concept in the financial literature. Kou (2002) proposed a double exponential jump diffusion (DEJD) model. In the DEJD model, the jumps are generated by a single Poisson process, and the upside and downside jump magnitudes are drawn from two independent exponential distributions. Later Ramezani and Zeng (2007) posited a Pareto-Beta jump-diffusion (PBJD) model in which the jumps are generated by two independent Poisson processes and the jump magnitudes are drawn from a Pareto distribution for upside jumps and a Beta distribution for downside jumps. In DEJD and PBJD, the distributions are univariate and the relationship with the market is not defined as in Camara's (2009) model. But, unlike Camara, they are able to restrict all positive jumps to be positive and all negative jumps to be negative. Since Camara's model is based on normality, it only restricts the mean of upside jumps to be positive and the mean of downside jumps to be negative.

Multivariate Mixed JD Process with Systematic Jumps

The JD processes that are described in the above subsections were univariate processes of asset prices. In these models, the influence of aggregate consumption[6] on asset price is not defined and so they implicitly assume non-systematic jumps. Camara's (2009) two counters of jumps model is a joint JD process of asset price and aggregate consumption is

[4] $B_s(T)$ and $M(T)$, $B_s(T)$ and $N(T)$, $B_s(T)$ and $Y_{su,i}$, $B_s(T)$and $Y_{sd,j}$, $M(T)$ and $Y_{su,i}$, $N(T)$ and $Y_{sd,j}$ are independent. In general the continuous components $B(T)$, the discrete components $M(T)$ and $N(T)$, and the jump magnitudes are independent. The magnitudes of jumps $\left(Y_{su,i}\text{ and }Y_{sd,j}\right)$ are allowed to be correlated. These assumptions are mentioned in subsequent models.

[5] The sample covariance of two variables X and Y each with sample size *n* is $cov(X,Y) = \sum_i (x_i - \bar{x})(y_i - \bar{y})/n$.

[6] The literature uses the term aggregate consumption and often calls these consumption-based asset pricing models. For estimation, consumption is replaced with a stock index. In our model, we use a commodity futures index rather than a stock index.

$$(6)\quad S_T = \exp\left(\ln(S_0) + \eta_s T - \frac{\sigma_s^2}{2}T + \sigma_s B_s(T) + \sum_{i=1}^{M(T)} Y_{su,i} + \sum_{j=1}^{N(T)} Y_{sd,j}\right)$$

$$(7)\quad C_T = \exp\left(\ln(C_0) + \eta_c T - \frac{\sigma_c^2}{2}T + \sigma_c B_c(T) + \sum_{i=1}^{M(T)} Y_{cu,i} + \sum_{j=1}^{N(T)} Y_{cd,j}\right)$$

where $M(T)$ is the Poisson counter process attributed to the incidence of upside jumps with intensity λ, $N(T)$ is the Poisson counter process attributed to the incidence of downside jumps with intensity parameter δ, C_0 is the current level of consumption, $B_c(T)$ is the consumption Brownian motion, η_c is the instantaneous expected growth rate of consumption without any jumps, σ_c is the variance of consumption without any jumps, and $Y_{cu,i}$ and $Y_{cd,j}$ are the upside and downside jump magnitudes in aggregate consumption. In this model, the aggregate consumption Brownian motion and asset price Brownian motion are correlated. The model also allows jump components to be correlated[7].

Restricting Camara's (2009) model to a single counter of jumps achieves the JD economy of Amin and Ng (1993). In their model, the jumps in the asset price are correlated with the jumps in aggregate consumption. The arrival of information in both series is defined by a single Poisson counting process $M(T)$ with intensity λ and the number of jumps in both series is the same. The joint mixed JD process with a single counter of jumps is

$$(8)\quad S_T = \exp\left(\ln(S_0) + \eta_s T - \frac{\sigma_s^2}{2}T + \sigma_s B_s(T) + \sum_{i=1}^{M(T)} Y_{s,i}\right)$$

$$(9)\quad C_T = \exp\left(\ln(C_0) + \eta_c T - \frac{\sigma_c^2}{2}T + \sigma_c B_c(T) + \sum_{i=1}^{M(T)} Y_{c,i}\right)$$

Aggregate consumption growth is not directly observable and is difficult to estimate. To bypass this, an index of market prices is used instead of consumption growth (Amin and Ng 1993). In this paper the CRB index of futures prices is used to represent the market. The joint mixed density function, defined as an extension of univariate distribution is

$$(10)\quad f_{r,m}(r,m) = \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \frac{1}{2\pi\sqrt{\left(\sigma_f^2+i\gamma_f^2\right)*\left(\sigma_m^2+i\gamma_m^2\right)}\sqrt{(1-r^2)}}\left(\exp\left[-\frac{Z}{2(1-r^2)}\right]\right)$$

$$(10a)\quad Z = \frac{\left(r_t-\mu_f-i\alpha_f\right)^2}{\left(\sigma_f+i\gamma_f\right)^2} - \frac{2r\left(r_t-\mu_f-i\alpha_f\right)(m_t-\mu_m-i\alpha_m)}{\left(\sigma_f+i\gamma_f\right)*(\sigma_m+i\gamma_m)} + \frac{(m_t-\mu_m-i\alpha_m)^2}{(\sigma_m+i\gamma_m)^2}$$

$$(10b)\quad r = \frac{\left(\rho_{fc}\sigma_f\sigma_m + \rho_{jp}\gamma_f\,\gamma_m\,i\right)}{\sqrt{\left(\sigma_f^2+i\gamma_f^2\right)(\sigma_m^2+i\gamma_m^2)}}$$

where μ_m and σ_m^2 are the instantaneous mean and variance of market returns without any jumps, α_m is the mean and γ_m^2 is the variance of jumps in the market returns, $Z/(1-r^2)$ is the square

[7] For more details of (auto)correlations between the jumps, see Camara (2009)

of Mahalanobis distance[8] of the multivariate distribution, r is the correlation coefficient between the two returns, ρ_{fc} is the correlation between returns conditional on zero jumps, and ρ_{jp} is the correlation between the jumps in both returns.

Multivariate Mixed JD Process with Systematic and Non-systematic Jumps

The generic one factor model that generalizes the popular CAPM model can be stated as $r_i = \alpha_i + \beta_i r_0 + \epsilon_i$, where i={1,…,N}, r_i is the returns on the i^{th} asset, r_0 is the systematic risk factor, and ϵ_i is the idiosyncratic risk that is uncorrelated with r_0.[9] Todorov and Bollerslev (2010) separated the systematic risk factor associated with continuous (r_0^c) and discontinuous (r_0^d) price moves. The extended one factor model can be stated as $r_i = \alpha_i + \beta_i^c r_0^c + \beta_i^d r_0^d + \epsilon_i$, i={1,…,N } where β_i^c and β_i^d represent the systematic risk attributable to continuous and discontinuous price moves, respectively. This paper introduces a new diffusion process that separates the systematic risk and non-systematic risk associated with jumps in futures prices. In this model, the uncorrelated jump represents only non-systematic discontinuous price moves whereas in Todorov and Bollerslev's (2010) model ϵ_i represents the idiosyncratic risk of both continuous and discontinuous price moves. The joint diffusion process that explains systematic jump risk is:

$$(11)\quad S_T = \exp\left(\ln(S_0) + \eta_s T - \frac{\sigma_s^2}{2} T + \sigma_s B_s(T) + \sum_{j=1}^{M(T)} Y_{sc,j} + \sum_{k=1}^{K(T)} Y_{su,k}\right)$$

$$(12)\quad C_T = \exp\left(\ln(C_0) + \eta_c T - \frac{\sigma_c^2}{2} T + \sigma_c B_c(T) + \sum_{j=1}^{M(T)} Y_{c,j}\right)$$

where $Y_{sc,j}$ represents the jump in asset price that is correlated with the market jumps $(Y_{c,j})$, the Poisson counting process that is attributed to the incidence of correlated jumps is $M(T)$ with intensity θ. The non-systematic jump magnitude in the asset prices ($Y_{su,k}$) is modeled as the second Poisson counting process $K(T)$ in the futures prices with an intensity parameter ω. The correlated jump and uncorrelated jump are analogous to r_0^d and ϵ_i respectively. Hence, in equation (11), it is assumed that the magnitude of systematic and non-systematic asset price jumps are not correlated $\left(\mathrm{cov}\left[\sum_{j=1}^{M(T)} Y_{sc,j}, \sum_{K=1}^{K(T)} Y_{su,k}\right] = 0\right)$. The corresponding joint mixed bivariate normal distribution is

$$(13)\quad f_{r,m}(r,m) = \sum_{j=1}^{\infty} \frac{e^{-\theta}\theta^j}{j!} \sum_{k=1}^{\infty} \frac{e^{-\omega}\omega^k}{k!}$$

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[8] If $\boldsymbol{y}$ has a multivariate normal distribution with covariance matrix $\boldsymbol{\Sigma}$ and mean vector $\boldsymbol{\mu}$, the density is given by $g(\boldsymbol{y}) = \frac{1}{(2\pi)^p|\boldsymbol{\Sigma}|^{1/2}}\, \boldsymbol{e^{-(y-\mu)'|\Sigma|^{-1}(y-\mu)/2}}$. The exponent term $(\boldsymbol{y} - \boldsymbol{\mu}'|\boldsymbol{\Sigma}|-1\ \boldsymbol{y}-\boldsymbol{\mu}$ is the squared generalized distance from $\boldsymbol{y}$ to $\boldsymbol{\mu}$ otherwise known as Mahalanobis distance (Rencher, 2002).

[9] For more details see Todorov and Bollerslev (2009).

$$\frac{1}{2\pi\sqrt{\left(\sigma_f^2 + j\gamma_{fc}^2 + k\gamma_{fu}^2\right)(\sigma_m^2 + j\gamma_m^2)\sqrt{(1-r^2)}}}\ exp\left[-\frac{Z}{2(1-r^2)}\right]$$

$$(13a)\quad Z = \frac{\left(x_f - \mu_f - j\alpha_{fc} - k\alpha_{fu}\right)^2}{\left(\sigma_f + j\gamma_{fc} + k\gamma_{fu}\right)^2} - \frac{2r\left(x_f - \mu_{fc} - j\alpha_{fc} - k\alpha_{fu}\right)(x_m - \mu_m - j\alpha_m)}{\left(\sigma_f + j\gamma_f + k\gamma_{fu}\right)(\sigma_m + j\gamma_m)} + \frac{(x_m - \mu_m - j\alpha_m)^2}{(\sigma_m + j\gamma_m)^2}$$

$$(13b)\quad r = \frac{\rho_{fc}\sigma_f\sigma_m + \rho_{jp}\gamma_{fc}\gamma_m j}{\sqrt{\left(\sigma_f^2 + j\gamma_{fc}^2 + k\gamma_{fu}^2\right)}\sqrt{(\sigma_m^2 + j\gamma_m^2)}}$$

where α_{fc} and α_{fu} are means, γ_{fc} and γ_{fu} are the variance of correlated and uncorrelated jumps. The betas associated with continuous price moves and discontinuous price moves are $\beta_{fc} = \rho_{fc}\left(\sigma_f/\sigma_m\right)$ and $\beta_{jp} = \rho_{jp}\left(\gamma_{fc}/\gamma_m\right)$ respectively. The likelihood expressions for the distribution functions are given in an Appendix.

Data and Procedure

Summary statistics were estimated and normality tests were performed for both returns series. The parameters of four increasingly general JD processes were estimated using numerical maximization of likelihood functions. Two univariate mixed normal distributions are estimated separately for each returns series with single and two counters of jumps. Likelihood ratio (LR) tests are used to select among nested models. Finally, the parameters of a joint mixed bivariate distribution with correlated single counter of jumps in both returns, and the model with separate systematic and non-systematic jumps in wheat futures prices are also estimated. Proc NLMIXED in SAS is used to estimate the parameters (SAS 2009). To maximize the likelihood function, the infinite sum has to be truncated (Jorion 1998). In this paper, the infinite sum is truncated at ten. An extensive Monte Carlo simulation was performed to ensure accuracy and reliability of statistical programs and procedures. Data were simulated (100,000 observations) from the JD models with known assumed parameter values. After the likelihood optimization, the resulting values are nearly identical to the assumed values.

The Kansas City Board of Trade (KCBT 2010) daily settlement wheat futures prices were used for a period of six years (January 2003 to December 2008). Five wheat futures contract maturity months (March, May, July, September, and December) are traded. A time-series dataset is constructed with the daily returns of nearby futures contracts. Rollover is done on the twentieth calendar day of the month prior to delivery. From the twenty-first calendar day onwards, the return of the next contract is in included in the dataset. This procedure reduces the maturity effect by only including contracts close to maturity. It also avoids the delivery period where position limits are removed and markets become thin. Differencing is performed before splicing to avoid creating outliers at the rollover.

A synchronous data set of the Thomson Reuters/Jefferies CRB index is constructed for the same time period. The CRB index is composed of four groups of nineteen components that include petroleum products, metals, and agricultural commodities. In the current study, CRB index futures represent the 'commodity market' (aggregate consumption) and the logarithmic price relatives as market returns (aggregate consumption growth). Wheat, with an index weight of 1%, is in group IV along with nickel, lean hogs, orange juice, and silver. The six expiration months traded are January, February, April, June, August, and November. The sixth calendar day of the month prior to the maturity month is selected as the rollover day. The differencing to calculate returns is performed before splicing the data. Both the CRB and wheat series include 1,380 observations of daily prices so as to provide enough degrees of freedom to use tests that are asymptotically valid. To reduce scaling problems, differences in daily prices are multiplied by 100 and expressed in percentage terms.

Empirical Results

Table 1 presents summary statistics and tests of normality. The positive excess kurtosis in both series indicates the presence of fat tails. The skewness of the wheat futures price differences series is close to zero while the CRB index shows considerable negative skewness.

The normality tests (Kolmogorov-Smirnov D statistic and Shapiro-Wilk W statistic) reject the null hypotheses at conventional levels of significance and make it clear that the returns are not normally distributed. The estimates of second and third moments and results of normality tests suggest that a JD process can provide a better fit for the returns than a normal distribution.

Table 1. Summary Statistics and Normality Tests for the Returns

Parameters	Wheat Futures	CRB Index
Minimum change in daily value	-8.451	-6.406
Maximum change in daily value	7.778	5.429
Mean change in daily value	0.023 [a]	0.009
Variance	3.636	1.049
Skewness	0.049	-0.755
Excess kurtosis	1.735	5.137
Kolmogorov-Smirnov D statistic	0.045	0.076
	(<0.01) [b]	(<0.01)
Shapiro-Wilk W statistic	0.981	0.932
	(<0.0001)	(<0.0001)

[a] The returns are expressed in percentages and the number of observations is 1,380.The beginning date of the data series is January 2003 and the ending date is December 2008.
[b] P-values are in the parentheses.

The parameter estimates and standard errors of jump diffusion processes with a single counter of jumps (equations (2) and (3)) are displayed in table 2. The jump intensity parameter is significant in both returns. The parameter estimate of jump intensity ($\lambda = 0.283$) indicates the presence of jumps in wheat futures returns that occur approximately once in every 4 business days ($\lambda^{-1} = 3.53$). As the jump component is defined using a single counter of jumps,

Table 2. Parameter Estimates of the Distribution with Single Counters of Jumps

Parameter	Symbol	Wheat Futures	CRB Index
Mean returns	μ	0.004 [a]	0.082 **
		(0.060) [b]	(0.024)
Volatility of returns	σ^2	2.001**	0.535**
		(0.235)	(0.036)
Intensity of jumps	λ	0.283**	0.129**
		(0.108)	(0.031)
Mean of jumps	α	0.066	-0.564**
		(0.199)	(0.204)
Volatility of jumps	γ^2	5.820**	3.710**
		(1.665)	(0.793)
Loglikelihood value		2804.73	1854.27
Bayesian information criterion		5645.60	3744.70

[a] The day to day changes in value are expressed in percentages.
[b] Standard errors are in the parentheses
* significant at 5% level
** significant at 1% level

the jump magnitude can be either positive or negative. The sample path of wheat futures shows the presence of frequent jumps (every 3.5 days) with a high variance and a mean not significantly different than zero. The volatility associated with the jump component (5.82%) is higher than the diffusion component (2%) in wheat futures returns. In CRB index futures returns, all parameter estimates are statistically significant. Jumps in the CRB index occur approximately once in 8 days ($\lambda^{-1} = 7.75$) and are less frequent compared to wheat futures. With index returns the mean of the jumps is negative ($\alpha = -0.56\%$), which is expected in order to model the negative skewness found in index returns.

The parameter estimates of the normal distribution with two counters of jumps (equations (4) and (5)) are presented in Table 3. A likelihood ratio (LR) test fails to reject the null hypothesis of a single jump versus the alternative of two counters of jumps for wheat futures returns ($\chi^2_{(4)}$ statistic is 1.05 and p-value is 0.9) and CRB index futures returns ($\chi^2_{(4)}$ statistic is 0.66 and p-value is 0.95).

Table 3. Parameter Estimates of the Distribution with Two Counters of Jumps

Parameter	Symbol	Wheat Futures	CRB index
Mean of returns	μ	-0.025 [a] (0.242) [b]	0.082** (0.024)
Volatility of returns	σ^2	2.292** (0.580)	0.402** (0.097)
Intensity of upside jumps	λ	0.063 (0.255)	0.593 (0.493)
Intensity of downside jumps	δ	0.032* (0.014)	0.102** (0.034)
Mean of upside jumps	α_u	3.015** (8.010)	0.000 -
Mean of down side jumps	α_d	-4.467** (0.682)	-0.695* (0.277)
Volatility of upside jumps	γ_u^2	2.472 (9.019)	0.304 (0.197)
Volatility of downside jumps	γ_d^2	0.0003 (0.064)	4.142 (2.421)
Covariance of jumps	v	0.028 (2.856)	0.002 (2.396)
Loglikelihood value		2803.68	1853.61
Bayesian information criterion		5672.40	3772.30

[a] The returns are expressed in percentages.
[b] Standard errors are in the parentheses.
* significant at 5% level.
** significant at 1% level.

Table 4 presents the parameter estimates of bivariate normal distribution (equations (8), (9), and (10)) with a single counter of jumps in wheat price futures and CRB index futures. All parameter estimates except the means of the diffusion component and the jump component are statistically significant. The joint model provides frequent jumps in returns $(\lambda = 0.158 \text{ and } \lambda^{-1} = 6.33)$ that occur once in 6 days. The intensity estimate is close to the estimate of univariate single counter of JD process in CRB index futures. During the study period the price of wheat (mean) is estimated as \$5.06 per bushel. It can be estimated that one standard deviation jumps are 14 cents bushel^{-1} and two standard deviation jumps are 29 cents bushel^{-1}. These estimates are reasonable and are within the price limits (30 cents bushel^{-1}).

As per equations (8) and (9), a jump is identified only when there is a simultaneous price movement of extraordinary magnitude in both return series. It appears from the mean of jumps in wheat futures $(\alpha_f = -0.133)$ and CRB index futures $(\alpha_m = -0.37)$ that the jumps are mostly crashes. The estimate of correlation between the diffusion components of wheat futures and CRB index futures returns $(\rho_{fc} = 0.335)$ shows less correlation than the jump components $(\rho_{jp} = 0.602)$. The estimates of beta show that the estimated systematic risk associated with jump components $(\beta_{jp} = 0.945)$ is higher than the estimated systematic risk associated with continuous components $(\beta_{fc} = 0.718)$. The result of a Wald t-test, however, indicates that the difference in the estimated betas is not statistically significant (t-value = 1.46, p-value = 0.145).

Table 4. Parameter Estimates of Bivariate Distribution with Single Counters of Jumps

Parameter	Symbol	Estimate
Mean of wheat futures return	μ_f	0.044[a] (0.052)[b]
Mean CRB index futures return	μ_m	0.068** (0.024)
Mean of wheat futures jump	α_f	-0.133 (0.272)
Mean of CRB index jump	α_m	-0.370* (0.155)
Volatility of wheat futures returns	σ_f^2	2.383** (0.144)
Volatility of CRB index futures	σ_m^2	0.519** (0.032)
Intensity of jumps	λ	0.158** (0.041)
Volatility of wheat futures jump	γ_f^2	7.948** (1.500)
Volatility of CRB index futures jump	γ_m^2	3.221** (0.61)
Correlation between futures and CRB index	ρ_{fc}	0.335** (0.031)
Correlation between jumps	ρ_{jp}	0.602** (0.064)
Beta of continuous components	β_{fc}	0.718** (0.070)
Beta of discontinuous (jump)components	β_{jp}	0.945** (0.123)
Loglikelihood value		4522.11
Bayesian information criterion		9123.8

[a] The returns are expressed in percentages.
[b] Standard errors are in the parentheses.
* significant at 5% level. ** significant at 1% level.

Table 5 discusses the estimated bivariate distribution with added non-systematic jumps in wheat futures returns. The magnitude of mean and variance of the continuous component changed with an additional jump in wheat futures while they remained almost the same for the CRB index futures.

The intensity of the correlated jump ($\theta = 0.13$ and $\theta^{-1} = 7.69$) is close to the intensity of the jump in the bivariate single counter of jumps model in Table 4. The intensity of correlated jumps is highly significant and reveals that systematic jumps occur once every 8 days. The correlated jumps are mostly crashes with a mean of -0.42% in wheat futures and -0.46% in CRB index futures. There is a wide gap between the correlation coefficient of the continuous component ($\rho_{fc} = 0.419$) and the jump component ($\rho_{jp} = 0.653$). In terms of wheat futures prices, correlated jumps of one and two standard deviations are 13 cents bushel^{-1} and 27 cents bushel^{-1} respectively. In the case of uncorrelated jumps, it can be estimated that the jumps are of smaller magnitudes (6 cents bushel^{-1} and 12 cents bushel^{-1} respectively). The additional non-systematic jump in the wheat futures made no significant change in the magnitudes of beta estimates ($\beta_{jp} = 0.891$ and $\beta_{fc} = 0.725$). The t-test indicates that the difference in the estimated betas is not statistically significant (p-value = 0.145).

Table 5. Parameter Estimates of Bivariate Distribution with Systematic and Non-systematic Risk

Parameter	Symbol	Estimate
Mean of wheat futures return	μ_f	-0.098[a] (0.132)[b]
Mean CRB index futures return	μ_m	0.069** (0.023)
Intensity of correlated jumps	θ	0.130** (0.027)
Intensity of uncorrelated jumps	ω	0.709 (0.420)
Mean of correlated wheat futures jump	α_{fc}	-0.417 (0.364)
Mean of uncorrelated wheat futures jump	α_{fu}	0.247 (0.171)
Mean of CRB index futures jump	α_m	-0.464* (0.190)
Volatility of wheat futures returns	σ_f^2	1.597** (0.357)
Volatility of CRB index futures returns	σ_m^2	0.534** (0.033)
Volatility of correlated wheat futures jump	γ_{fc}^2	7.067 (1.627)
Volatility of uncorrelated wheat futures jump	γ_{fu}^2	1.472** (0.614)
Volatility of CRB index futures jump	γ_m^2	3.788** (0.765)
Correlation between wheat futures and CRB index	ρ_{fc}	0.419** (0.056)
Correlation between jumps	ρ_{jp}	0.653** (0.241)
Beta of continuous components	β_{fc}	0.725 (0.069)
Beta of discontinuous (jump)components	β_{jp}	0.891 (0.126)
Loglikelihood value		4518.02
Bayesian information criterion		9137.20

[a] The returns are expressed in percentages
[b] Standard errors are in the parentheses
* significant at 5% level ** significant at 1% level

A likelihood ratio test is employed to select among the nested models. The LR test fails to reject ($\chi^2_{(3)}$ statistic is 4.09 and p-value is 0.25) the null hypothesis of imposed restrictions $(\omega = 0, \alpha_{fu} = 0, \text{and } \gamma_{fu} = 0)$ and showed the bivariate distribution with a single counter of jumps (Amin and Ng, 2003) in both the wheat market and CRB index fits the data better than the model with two counters of jumps in the wheat futures market. The overall results including the summary statistics and the univariate models with single and two counters of jumps indicate that wheat futures returns do not need its own jump component. The estimated skewness and excess kurtosis of the futures returns do not deviate as much from normality as does the CRB index.

Conclusions

Camara's (2009) two counters of jumps model generalizes Merton's (1976) JD process that incorporates discontinuous jumps in asset price and the Amin and Ng (1993) model. The current paper contributes to the existing literature on the distribution of changes in futures prices by employing a mixed bivariate normal distribution and estimates the parameters using wheat futures prices and CRB index futures. The paper also extends Camara's (2009) model by defining jumps as systematic and non-systematic. The empirical analysis shows that on average a jump occurs once every 6 days. In terms of wheat prices, one standard deviation jumps are 14 cents per bushel and two standard deviation jumps are 29 cents per bushel and are within the price limits. The high correlation between the jumps in wheat futures and CRB index futures indicates the presence of systematic risk associated with jumps. Camara's (2009) generalization of two counters of jumps based on the jump magnitudes is not suitable for wheat futures data. Camara's (2009) distinction of upside and downside jumps was implemented by imposing bounds. As the bounds are active, Camara's (2009) distinction between upside and downside jumps does not match these data. The magnitudes of the betas associated with the continuous and jump components were not statistically different in both joint JD processes. In general, the results support Amin and Ng (1993) joint JD process with a single counter of jumps.

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Appendix: Maximum Likelihood Functions

The appendix describes the maximum likelihood estimation method. The notation closely follows Jorion (1988). Five different sets of density function parameters are estimated. If $r_1, r_2, \ldots, r_T$ are continuous random variables that represent the normally distributed wheat futures returns $r_T \sim N(\mu_f, \sigma_f^2)$ then the logarithm of the likelihood function as a function of parameter vector $\theta = (\mu_f, \sigma_f^2)$ for a normal distribution can be written as

$$(A1) \quad l(\theta) = -\frac{T}{2}\ln(2\pi) + \sum_{t=1}^{T} ln\left[\frac{1}{\sqrt{\sigma_f^2}} \exp\left(-\frac{(f_t - \mu_f)^2}{2\sigma_f^2}\right)\right]$$

With a single counter of jumps, the log likelihood function for the mixed jump-diffusion process (equation 3) with parameter vector $\theta = (\mu_f, \sigma_f^2, \lambda, \alpha_f, \gamma_f^2)$ can be written as

$$(A2) \quad l_i(\theta) = -T\lambda - \frac{T}{2}\ln(2\pi) + \sum_{t=1}^{T} \ln\left[\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{1}{\sqrt{\sigma_f^2 + i\gamma_f^2}} \exp\left(-\frac{(f_t - \mu_f - i\alpha_f)^2}{2(\sigma_f^2 + i\gamma_f^2)}\right)\right]$$

With two counters of jumps, the logarithm of the likelihood function for the mixed jump-diffusion process (equation 5) with parameter vector $\theta = (\mu_f, \sigma_f^2, \lambda, \delta, \alpha_{fu}, \alpha_{fd}, \gamma_{fu}^2, \gamma_{fd}^2, \upsilon)$ is

$$(A3) \quad l_{ij}(\theta)$$
$$= T(\lambda + \delta) - \frac{T}{2}\ln(2\pi)$$
$$+ \sum_{t=1}^{T} \ln\left[\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \sum_{j=0}^{\infty} \frac{\delta^j}{j!} \frac{1}{\sqrt{\sigma_f^2 + i\gamma_{fu}^2 + j\gamma_{fd}^2 + 2\min(i,j)\, \upsilon_f}} \exp\left(-\frac{(f_t - \mu_f - (i\alpha_{fu} + j\alpha_{fd}))^2}{2(\sigma_f^2 + i\gamma_{fu}^2 + j\gamma_{fd}^2 + 2\min(i,j)\, \upsilon_f)}\right)\right]$$

With a bivariate normal distribution, logarithm of the likelihood function for the mixed jump-diffusion process (equation 6) with parameter vector $\theta = (\mu_f, \sigma_f^2, \lambda, \alpha_f, \gamma_f^2, \sigma_m^2, \gamma_m^2, r)$ can be written as

$$(A4) \quad l_j(\theta) = -T\lambda - T\ln(2\pi) + \sum_{t=1}^{T} \ln\left[\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{1}{\sqrt{(\sigma_f^2 + i\gamma_f^2)(\sigma_m^2 + i\gamma_m^2)\sqrt{(1 - r^2)}}} \exp\left(-\frac{Z}{2(1 - r^2)}\right)\right]$$

$$Z = \frac{(x_f - \mu_f - i\alpha_f)^2}{(\sigma_f + i\gamma_f)^2} - \frac{2r(x_f - \mu_f - i\alpha_f)(x_m - \mu_m - i\alpha_m)}{(\sigma_f + i\gamma_f)(\sigma_m + i\gamma_m)} + \frac{(x_m - \mu_m - i\alpha_m)^2}{(\sigma_m + i\gamma_m)^2}$$

With a bivariate normal distribution that has separate systematic and non-systematic jump risk, the logarithm of likelihood function for the mixed jump-diffusion process (equations 9, 10, and 11) with parameter vector $\theta = (\mu_f, \mu_c, \sigma_f^2, \sigma_m^2, \theta, \omega, \alpha_{fc}, \alpha_{fc}, \alpha_c, \gamma_{fc}^2, \gamma_{fu}^2, r)$ can be written as

$$(A5) \quad l_j(\theta) =$$
$$-T(\theta + \omega) - T\ln(2\pi) + \sum_{t=1}^{T} \ln\left[\sum_{i=0}^{\infty} \frac{\theta^i}{i!} \sum_{k=0}^{\infty} \frac{\omega^k}{k!} \frac{1}{2\pi\sqrt{(\sigma_f^2 + j\gamma_{fc}^2 + k\gamma_{fu}^2)(\sigma_m^2 + j\gamma_m^2)\sqrt{(1 - r^2)}}} \exp\left[-\frac{Z}{2(1 - r^2)}\right]\right]$$

$$Z = \left[\frac{(x_f - \mu_f - j\alpha_{fc} - k\alpha_{fu})^2}{(\sigma_f + j\gamma_{fc} + k\gamma_{fu})^2} - \frac{2r(x_f - \mu_{fc} - j\alpha_{fc} - k\alpha_{fu})(x_m - \mu_m - j\alpha_m)}{(\sigma_f + j\gamma_f + k\gamma_{fu}) * (\sigma_m + j\gamma_m)} + \frac{(x_m - \mu_m - j\alpha_m)^2}{(\sigma_m + j\gamma_m)^2}\right];$$

$$r = \frac{\rho_{fc}\sigma_f\sigma_m + \rho_{jump}\gamma_f\gamma_m j}{\sqrt{(\sigma_f^2 + j\gamma_{fc}^2 + k\gamma_{fu}^2)} * \sqrt{(\sigma_m^2 + j\gamma_m^2)}}$$