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Practitioners Abstract

Schwartz's (1997) two-factor model is generalized to allow for mean reversion in spot prices. Our modeling also acknowledges that commodities exhibit seasonality patterns in the spot price. A Bayesian MCMC algorithm is developed to estimate our model. Estimation of the Schwartz model is done by imposing appropriate restrictions to our model. Estimation results are obtained based on monthly observations of soybean futures prices and options prices from the Chicago Board of Trade over period from January 1978 to January 2010. We empirically estimate and compare our model with the Schwartz model, and show how the assumption of mean reversion in spot prices and seasonality affect the prediction of futures prices and option premium with short and long times to maturity.

Keywords
Commodity markets, mean reversion, seasonality, futures and option pricing

1 Introduction

A variety of models have been developed to capture the stochastic behavior of commodity prices. A number of commodity derivatives pricing formulas, such as futures, swaps and options, are proposed based on these models. For example, Schwartz (1997) develops and empirically compares three models. In his two-factor model, spot prices and the convenience yield rate are employed as the latent state variables. Miltersen and Schwartz(1998) and Hillard and Reis (1998) propose closed-form futures and option-pricing formulas. They incorporate mean reversion in the convenience yield. However, the price level is assumed to trend rather than revert to a long-term mean. Therefore, such models most likely are relevant to exhaustible commodity markets such as gold and oil, where Hotelling's Principle might be expected to hold. A number of studies report evidence of mean reversion in commodity prices. (Peterson, Ma, and Ritchie(1992); Allen, Ma, and Pace(1994); Walburger and Foster(1995)). We argue that mean reversion in spot price is a key feature of agricultural commodity markets such as grain or livestock, and here we can expect mean reversion in both the level of prices and the convenience yield. For example, when prices are relatively high, supply will increase, which will in turn put a downward pressure on prices. On the other hand, when prices are relatively low, supply will decrease, which will induce prices to increase. We generalize Schwartz's (1997) two-factor model by allowing for mean reversion in the spot price level. We show that the Schwartz model is a special case of ours.

Seasonality is known to be an empirical characteristic of most commodity markets. It is especially important for agricultural commodities with a seasonal production pattern. Seasonality is introduced into our model by allowing the parameters in the drift terms to a be periodical function of calendar time. The evaluation of futures and option pricing expressions can be reduced to the problem of solving ordinary differential equations (Duffie, Pan, and Singleton). Adding seasonality into the model makes the solution more complicated. However, we still get closed-form expressions, which greatly facilitate empirical work.

We develop a Bayesian MCMC algorithm to estimate our model. At each sample date, we observe a panel of futures prices and option premiums which are related to the latent variables through the futures and option pricing formulas. Following the ideas in Chen and Scott (1993), we assume that all but two futures contract prices are observed with measurement error. The latent value of the state-variables can be filtered out at each sample date using the futures pricing formula by inversion based on the two futures contract prices which are observed without error. The empirical results support our postulation that spot price in agricultural commodity markets are mean reverting. Seasonality patterns are clear in our estimation. We show how the basic assumption of mean-reverting spot prices will affect the model's prediction about the price of commodity derivatives with short and long times to maturities by comparing Schwartz' model and our model.

The rest of this paper is organized as follows. In section 2, we generalize Schwartz' two-factor model. Seasonality is also introduced into the advocated model. In section 3, futures and option pricing formulas are derived. Section 4 describes our empirical model. The econometric analysis is performed in section 5. The last section concludes the paper.

2 Schwartz Model and Generalization

Schwartz advanced a path-breaking model of commodity prices. His fundamental insight is that commodities are characterized by convenience yields. He postulated that the convenience yield net of storage cost, c_t, follows the Ornstein-Uhlenbeck stochastic process:

$$dc = (\mu_c - k_c c)dt + \sigma_c dw_c \tag{1}$$

where μ_c/k_c is the long-run mean of the convenience yield, $k_c > 0$ is the convenience yield's speed of mean reversion, and dw_c is a Wiener process. However, Schwartz assumes that the process of commodity spot price is not mean reverting. It is assumed to behave as a geometric Brownian motion:

$$dS = \mu_s S dt + \sigma_s S dw_s \tag{2}$$

where dw_s is a Wiener process, and $dw_c dw_s = \rho_{sc} dt$. Define $x \equiv \ln(S)$. Application of Ito's Lemma yields the stochastic process for x,

$$dx = \mu_x dt + \sigma_x dw_x \tag{3}$$

where $\mu_x \equiv \mu_s - \sigma_s^2/2$, $\sigma_x \equiv \sigma_s$, $dw_x \equiv dw_s$, and $\rho_{xc} \equiv \rho_{sc}$. The expected rate of return to the commodity holder consists of the relative price change ($dS/S = dx$) plus the net convenience yield (c). In equilibrium, the expected rate of return to the commodity holder must equal the risk-free rate (r) plus the risk premium associated with stochastic process of x (λ), $\mu_x + c = r + \lambda$. Then the corresponding risk-neutral processes are:

$$\begin{aligned} dc &= (\mu_c - k_c c - \lambda_c)dt + \sigma_c dw_c^* \\ dx &= (r - c)dt + \sigma_x dw_x^* \end{aligned}$$

where λ_c is the market price for the risk associated with stochastic process of c, and dw_c^* and dw_x^* are the Wiener process under the equivalent martingale measure. Note that $dw_c^* dw_x^* = \rho_{xc} dt$. Schwartz derived futures prices based on this model. The corresponding option prices expression is obtained by Miltersen and Schwartz (1998), and Hilliard and Reis (1998).

Unlike Schwartz, the model proposed in our study allows the spot prices to revert to a long term mean,

$$dS_t = [u_s - k_s \ln(S_t)]S_t dt + \sigma_s S_t dw_s \tag{4}$$

Ito's Lemma yields the Ornstein-Uhlenbeck stochastic process for the logarithm of the spot prices:

$$\begin{aligned} dx_t &= (u_x - k_x x_t)dt + \sigma_x dw_x \\ &= k_x(u_x/k_x - x_t)dt + \sigma_x dw_x \end{aligned} \tag{5}$$

where $k_x \equiv k_s > 0$ is the speed of mean reversion and u_x/k_x is the long run mean.

A stylized fact of commodity markets is that convenience yields are positively associated with spot prices[1]. Hence, the convenience yield net of storage costs is postulated to consist of the following random function of the logarithm of the spot price:

$$c_t = y_t + k_x x_t \tag{6}$$

The first component of c is assumed to follow the Ornstein-Uhlenbeck stochastic process:

$$dy_t = k_y[u_y/k_y - y_t]dt + \sigma_y dw_y \tag{7}$$

where $dw_x dw_y = \rho_{xy} dt$.

2.1 Seasonality

So far, the models considered assumed that all parameters are constant through out the year. A feature of commodity markets, that differs from the markets for stocks, bonds, and other conventional financial assets, is that they exhibit seasonality. For example, prices for annual crops are high on the pre-harvest season and low at peak-harvest. To capture this feature, the periodicity in the corresponding parameters is represented by a truncated Fourier series. We add seasonality into the price by setting u_x in equation (5) to

[1] Typically, when a commodity is in relatively short supply, its price is high and its convenience yield is high as well.

be a periodical deterministic function of time.

$$dx_t = du_x(t)/k_x + k_x[u_x(t)/k_x - x_t]dt + \sigma_x dw_x \tag{8}$$

$$u_x(t) = u_{x,0} + \sum_{h=1}^{H}\left[u_{x,h,\cos}Cos(\frac{2\pi h}{s}t) + u_{x,h,\sin}Sin(\frac{2\pi h}{s}t)\right]$$

where s is the number of observations per year, H determines the number of terms in the sum, and $u_{x,h,\cos}$ and $u_{x,h,\sin}$ are seasonality parameters. Since monthly data are going to be employed to estimate the model, s is chosen to be 12. H is selected to be equal to 2. This choice is based on the Akaike Information Criterion (AIC); see, e.g., Harvey (1981). In this model setup, $u_x(t)/k_x$ will be the long-run mean of $x(t)$. Note that if $k_x = 0$, and u_x is a constant, (8) collapses to (3), that is, the geometric Brownian motion with expected drift u_x assumed by Schwartz.

The first component of convenience yield is also generalized to follow the Ornstein-Uhlenbeck stochastic process with seasonality:

$$dy_t = du_y(t)/k_y + k_y[u_y(t)/k_y - y_t]dt + \sigma_y dw_y \tag{9}$$

where $u_y(t)$ has the same functional form as $u_x(t)$. We can see that if $u_y(t)$ is a constant, then equation (9) is the same as equation (7). In equilibrium, the instantaneous expected total return to commodity holders must equal the risk-free rate plus the associated market price of risk:

$$\begin{aligned} E(ds_t/s_t + c_t) &= du_x(t)/k_x + [(u_x(t) - k_x x_t) + (y_t + k_x x_t)]dt \\ r + \lambda_x(t) &= (\frac{1}{k_x}\frac{du_x(t)}{dt} + u_x(t) - k_x x_t) + (y + k_x x_t) \\ &\Rightarrow (\frac{1}{k_x}\frac{du_x(t)}{dt} + u_x(t) - k_x x_t) - \lambda_x(t) = r - (y_t + k_x x_t) \end{aligned}$$

Similar to the expression for u_x, the risk premium is also assumed to be a periodical function of calendar time

$$\lambda_x(t) \equiv \lambda_{x,0} + \sum_{h=1}^{H}\left[\lambda_{x,h,\cos}Cos(\frac{2\pi h}{s}t) + \lambda_{x,h,\sin}Sin(\frac{2\pi h}{s}t)\right]$$

Therefore the risk-neutral process of dx_t may be written as:

$$dx_t = [r - (y_t + k_x x_t)]dt + \sigma_x dw_x^* \tag{10}$$

Let the market price for the y_t risk be $\lambda_y(t)$, then the risk-neutral process of dy_t is:

$$dy_t = (\frac{1}{k_y}\frac{du_y(t)}{dt} + u_y(t) - k_y y_t - \lambda_y(t))dt + \sigma_y dw_y^* \tag{11}$$

where $dw_x^* dw_y^* = \rho_{xy}dt$

$$\lambda_y(t) \equiv \lambda_{y,0} + \sum_{h=1}^{H}\left[\lambda_{y,h,\cos}Cos(\frac{2\pi h}{s}t) + \lambda_{y,h,\sin}Sin(\frac{2\pi h}{s}t)\right]$$

Setting $k_x = 0$, and $u_y(t)$ being a constant in (6) yields $c_t = y_t$, in which case our model becomes identical to Schwartz's (1997) two-factor model.

The risk-neutral processes (10) and (11) provide us the basic foundations to pricing futures and option contracts on commodity markets, which is done in the next section.

3 Futures and Option Pricing

Commodity prices and convenience yields are modeled in continuous-time as a system of stochastic differential equations in an affine term structure class. Affine models are tractable for asset pricing purposes. We rely on the traditional no-arbitrage approach to price commodity derivatives. The seasonality part makes the derivation more complicated. However, we still get closed-form solutions for the pricing formulas. The following two subsections show the process of valuation of commodity futures contracts and options written on commodity futures.

3.1 Futures Pricing

The risk-neutral process of the two latent variables defined in section (2.1) in the advocated model can be written as

$$\left[\begin{array}{c} dx_t \\ dy_t \end{array}\right] \sim N\left(\left[\begin{array}{c} r - k_x x_t - y_t \\ \frac{1}{k_y}\frac{du_y(t)}{dt} + u_y(t) - \lambda_y(t) - k_y y_t \end{array}\right] dt, \left[\begin{array}{cc} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{array}\right] dt^*\right)$$

let $\theta(t) \equiv \frac{1}{k_y}\frac{du_y(t)}{dt} + u_y(t) - \lambda_y(t) \equiv \theta_0 + \sum_{h=1}^{H}\left[\theta_{h,\cos}Cos(\frac{2\pi h}{s}t) + \theta_{h,\sin}Sin(\frac{2\pi h}{s}t)\right]$; where $\theta_0 \equiv u_{y,0} - \lambda_y(t)$; $\theta_{h,\cos} \equiv u_{y,h,\cos} + \frac{2\pi h}{s}u_{y,h,\sin}$; $\theta_{h,\sin} \equiv u_{y,h,\sin} - \frac{2\pi h}{s}u_{y,h,\cos}$.

Define $K_0 \equiv [r, \theta(t)]^T$, $K_1 \equiv \left[\begin{array}{cc} k_x & 1 \\ 0 & k_y \end{array}\right]$, $V \equiv \left[\begin{array}{cc} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{array}\right]$, $\mu_t \equiv [x_t, y_t]^T$. Write it into matrix form:

$$d\mu_t = (K_0 - K_1\mu_t)dt + V dw_t^*$$

Given that $\mu(t)$ follows an affine diffusion under the martingale measure, it is convenient to apply the method proposed by Duffie, Pan, and Singleton (DPS) (2000) to get a closed-form solution for the futures price as follows,

$$\begin{aligned} F(t,\tau) &= E_t^*[P(t+\tau)] \\ &= E_t^*\{\exp[\phi_0 + \phi^T \mu(t+\tau)]\} \\ &= \exp[\alpha(t) + \beta(t)\mu(t)] \\ &\Rightarrow X_{t,\tau} \equiv \ln(F(\mu(t), t, T)) = \alpha(t) + \beta(t)\mu(t) \end{aligned} \tag{12}$$

where $E_t^*[.]$ is the expectation operation under the risk-neutral probability measure. Since the first factor is the log of spot price, $\phi_0 = 0$, $\phi^T = [1, 0]$. $\alpha(t)$ and $\beta(t)$ need to satisfy the following ODEs

$$\begin{aligned} \beta^{'}(t) &= K_1^T \beta(t) \\ \alpha^{'}(t) &= -K_0(t) \cdot \beta(t) - \frac{1}{2}\beta^T(t) V \beta(t) \end{aligned}$$

with boundary conditions $\beta(T) = \phi,\ \ \alpha(T) = \phi_0$. The solution for $\alpha(t)$ and $\beta(t)$ is outlined in Appendix A.

3.2 Option Pricing

The process of solving for the option pricing expression is very similar to the one we propose to derive the futures price formula. Define $\tau_1 \equiv T_1 - t$. Let $C[F(t+\tau_1, T), K, t, T_1]$ denote the price at time t of a European call option expiring at $T_1 = t + \tau_1$ on a futures contract that expires at time $T \geq T_1$, with strike price K. The payoff of such an option at expiration date is $\max[F(t + \tau_1, T) - K, 0]$. Standard arguments can be applied to show that its price at time t is given by

$$C[F(t+\tau_1, T), K, t, T_1] = \exp[r(t - T_1)] E_t^*\{\max[F(t+\tau_1, T) - K, 0]\}$$

The moment generating function of the logarithm of futures prices at time T_1 under the equivalent martingale measure is defined by

$$M_{\ln[F(t+\tau_1,T)]} \equiv E_t^*\{\exp[z \ln(F(t+\tau_1, T))]\}$$

Using the futures price formula that we derived in the previous section to substitute for $\ln(F(t + \tau_1, T))$

$$\begin{aligned} M_{\ln[F(t+\tau_1,T)]}(z) &\equiv E_t^*\{\exp[z \ln(F(t+\tau_1, T))]\} \\ &= E_t^*\{\exp(z\alpha(T_1) + z\beta(T_1)\mu(T_1))\} \\ &= \exp\{z\alpha(T_1)\} E_t^*\{\exp(z\beta_1(T_1) x_{T_1} + z\beta_2(T_1) y_{T_1})\} \end{aligned}$$

The expectation term on the right-hand side is of the same form as equation (2.3) in DPS, so their method can be applied again. The underlying risk-neutral process does not change so the ODEs do not change. However, the boundary conditions are different. Let the solution to the expectation term be

$$\exp[A(t) + B(t)\mu(t)]$$

Then $A(\tau_1)$ and $B(\tau_1)$ satisfy the following ODEs

$$\begin{aligned} B^{'}(t) &= K_1^T B(t) \\ A^{'}(t) &= -K_0(t) \cdot B(t) - \frac{1}{2} B^T(t) V B(t) \end{aligned}$$

and boundary conditions,

$$\begin{aligned} B_1(T_1) &= z\beta_1(T_1),\ B_2(T_1) = z\beta_2(T_1) \\ A(T_1) &= 0 \end{aligned}$$

Solving this system of ODEs using the same strategy as we employed to solve for the futures price, we get the following results

$$B_1(t) = z\beta_1(t),\ B_2(t) = z\beta_2(t)$$

By definition of $\eta_1(t)$ and $\eta_2(t)$ from Appendix A,

$$A(t) = (\eta_1(t) - \eta_1(T_1))z + (\eta_2(t) - \eta_2(T_1))z^2$$

So, it yields the following expression for the moment generating function:

$$M_{\ln[F(T_1,T)]}(z) = \exp[\varpi(\mu(t), t, T_1, T)z + \frac{1}{2}\sigma(t, T_1, T)^2 z^2]$$

where

$$\begin{aligned} \varpi(t, T_1, T) &= \eta_1(T_1) + \eta_2(T_1) + \eta_1(t) - \eta_1(T_1) + \beta_1(t)x_t + \beta_2(t)y_t \\ &= \eta_2(T_1) + \eta_1(t) + \beta_1(t)x_t + \beta_2(t)y_t \\ \sigma(t, T_1, T)^2 &= 2[\eta_2(t) - \eta_2(T_1)] \end{aligned}$$

The specific form of the moment generating function reminds us that $\ln[F(t+\tau_1, T)]$ is distributed as a normal random variable with mean $\varpi(t, T_1, T)$ and variance $\sigma(t, T_1, T)^2$. In addition,

$$\begin{aligned} \exp[\varpi + \frac{1}{2}\sigma^2] &= \exp[\eta_2(\tau - \tau_1) + \eta_1(\tau) + \eta_2(\tau) - \eta_2(\tau - \tau_1) + \beta_1(\tau)x_t + \beta_2(\tau)y_t] \\ &= \exp[\eta_1(\tau) + \eta_2(\tau) + \beta_1(\tau)x_t + \beta_2(\tau)y_t] \\ &= \exp[\alpha(\tau) + \beta(\tau)\mu(t)] = F(t, T) \end{aligned} \tag{13}$$

After getting the probability density function of $\ln[F(t+\tau_1, T)]$, it is straightforward to derive the option price, (See appendix B)

$$E_t^*\{\max[F_{T_1} - K, 0]\} = \int_{\ln(K)}^{\infty} (F_{T_1} - K)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1})$$

The analytical solution for the price of the call option is:

$$C[F(t+\tau_1, T), K, t, T_1] = \exp[r(t - T_1)][F(\mu(t), t, T)N(d_1) - KN(d_2)] \tag{14}$$

where $N(\cdot)$ is the standard normal cumulative distribution function, $d_1 \equiv \frac{\ln[F(\mu(t),t,T)/K]+0.5\sigma(t,T_1,T)^2}{\sigma(t,T_1,T)}$, and $d_2 \equiv \frac{\ln[F(\mu(t),t,T)/K]-0.5\sigma(t,T_1,T)^2}{\sigma(t,T_1,T)}$.

Put-call parity can be used to get the analogous European put option price as follows,

$$\begin{aligned} P[F,K,t,T_1] &= \exp[r(t-T_1)]E_t^*\{\max[K-F(T_1,T),0]\} \\ &= \exp[r(t-T_1)]E_t^*\{\max[F(T_1,T)-K,0]+K-F(T_1,T)\} \\ &= \exp[r(t-T_1)]\{E_t^*[\max(F(T_1,T)-K,0]+K-E_t^*[F(T_1,T)]\} \\ &= C[F(t,T),K,t,T_1]+\exp[r(t-T_1)][K-F(t,T)] \end{aligned} \tag{15}$$

4 Empirical Model

In our model, we employ the log of the spot price and convenience yield as two latent state variables. Both of them are assumed to be mean reverting in the historical measure. The risk-neutral process of the two Gaussian factors can be written as:

$$\begin{aligned} d\mu_t &\sim N\left((K_0 - K_1\mu_t)dt,\ Vdt^*\right) \\ & \quad N\left(\begin{bmatrix} r-k_x x_t - y_t \\ \theta(t)-k_y y_t \end{bmatrix} dt,\ \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} dt^*\right) \end{aligned}$$

where $d\mu_t \equiv [dx_t, dy_t]^T$. Hence, the discrete-time counterpart of the historical evolution equation is

$$\Delta\mu_t = K_0 - K_1\Delta\mu_{t-1} + \Delta\lambda_t + \varepsilon_t,\ \varepsilon_t \sim N(\underline{0}_{(2\times1)}, V) \tag{16}$$

The futures pricing formula is outlined in section 3.1. $X_{t,\tau} \equiv \ln(F(\mu(t),t,T)) = \alpha(t)+\beta(t)\mu(t)$. Suppose we have a historical data set consisting of $M > 2$ series of (logarithms of) futures prices with M different time to maturity. Among the M futures contracts with distinct maturity dates, two of the prices are assumed to be perfectly correlated with the state variables μ_t, and the remaining $M-2$ are assumed to be observed with possibly heteroscedastic errors. Denote the two perfectly correlated futures prices with $X_t^\circ \equiv [X_{t,1}^\circ, X_{t,2}^\circ]^T$ and their maturities $[\tau_1^\circ, \tau_2^\circ]^T$ and the imperfectly correlated ones with $X_t^\cdot \equiv [X_{t,1}^\cdot, X_{t,2}^\cdot \cdots X_{t,M-2}^\cdot]^T$ and their maturities $[\tau_1^\cdot, \cdots \tau_{M-2}^\cdot]^T$. So,

$$\begin{bmatrix} X^{\circ}_{t,1} \\ X^{\circ}_{t,2} \end{bmatrix} = \begin{bmatrix} \alpha(\tau^{\circ}_1) \\ \alpha(\tau^{\circ}_2) \end{bmatrix} + \begin{bmatrix} \beta_1(\tau^{\circ}_1) & \beta_2(\tau^{\circ}_1) \\ \beta_1(\tau^{\circ}_2) & \beta_2(\tau^{\circ}_2) \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

$$\begin{bmatrix} X^{\cdot}_{t,1} \\ \vdots \\ X^{\cdot}_{t,M-2} \end{bmatrix} = \begin{bmatrix} \alpha(\tau^{\cdot}_1) \\ \vdots \\ \alpha(\tau^{\cdot}_{M-2}) \end{bmatrix} + \begin{bmatrix} \beta_1(\tau^{\cdot}_1) & \beta_2(\tau^{\cdot}_1) \\ \vdots & \vdots \\ \beta_1(\tau^{\cdot}_{M-2}) & \beta_2(\tau^{\cdot}_{M-2}) \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} e_{t,1} \\ \vdots \\ e_{t,M-2} \end{bmatrix}$$

$$\begin{bmatrix} e_{t,1} \\ \vdots \\ e_{t,M-2} \end{bmatrix} \sim N(\underline{0}_{((M-2)\times 1)}, \sigma_e^2 \Omega_{e,t})$$

Putting them into matrix form, we get,

$$X^{\circ}_t = \alpha^{\circ} + \beta^{\circ} \mu_t \tag{17}$$

$$X^{\cdot}_t = \alpha^{\cdot} + \beta^{\cdot} \mu_t + e_t \tag{18}$$

where $\underline{0}_{((M-2)\times 1)}$ is an $(M-2)$ vector of zeros, $\sigma_e^2 > 0$ is a scalar, and $\Omega_{e,t}$ is an $(M-2) \times (M-2)$ matrix, with i, jth element equal to $\rho^{|i-j|}$. $\alpha(\tau)$ and $\beta(\tau)$ are defined in section 2.1. Since the two latent factors are not observed, direct estimation of the historical evolution equation is not feasible. However, given equation (17), if the 2×2 matrix β° is invertible, it is easy to solve for the factors $\mu_t = (\beta^{\circ})^{-1}(X^{\circ}_t - \alpha^{\circ})$. In this way, the value of the state variables can be exactly filtered out at each sample date by inversion based on the two contract prices which are observed without error.

Similar to the imperfectly correlated futures prices, option prices are assumed to be observed with errors. According to equations (14) and (15), the empirical model is specified as,

$$\begin{aligned} C[F(\mu(t), t, T), K, t, T_1] &= \exp[r(t-T_1)][F(\mu(t), t, T)N(d_1) - KN(d_2)] + \varepsilon_{t,c} \\ P[F(\mu(t), t, T), K, t, T_1] &= \exp[r(t-T_1)][F(\mu(t), t, T)N(d_1) - KN(d_2)] \\ &\quad + \exp[r(t-T_1)][K - F(\mu(t), t, T)] + \varepsilon_{t,p} \end{aligned}$$

where $\varepsilon_{t,c} \sim N(0, \sigma_c^2)$, $\varepsilon_{t,p} \sim N(0, \sigma_p^2)$. We add a serially and cross-sectionally uncorrelated mean zero disturbance into the put and call option formula to take into account nonsimultaneity of the observation, errors in the data, and other potential sources of errors. We define the parameter set to be estimated as $\Phi \equiv \{V, \lambda_y, K_0, K_1, \rho, \sigma_e, \sigma_c, \sigma_p\}$.

5 Empirical Implementation

5.1 Description of the Data

Panel data of soybean futures prices are obtained from the Chicago Board of Trade (CBOT). Futures prices are observed monthly from January 1978 to January 2010. So, we have 385 observation dates in total. The futures prices involved are settlement prices at CBOT for the $15th$ calendar day of the month. If the $15th$ of the month is a holiday, the nearest trading day's settlement price is used. Since the longest maturity in the sample is 34 months, the ideal data set would consist of a panel of $385 \times 34 = 13090$ observations. The i, jth element of our data set is the price of the contract expires in j months at date i. However, futures for some maturities are not traded, because according to CBOT's regulation soybean futures can only mature on January, March, May, July, August, September, and November. In addition, data with far-away maturities are often missing because they are not traded. For example, on January 1980 only seven prices are observed. They are the $2nd$, $4th$, $6th$, $7th$, $8th$, $10th$,and $12th$ element of the $25th$ row of our data set, which correspond to the expiration dates of March, May, July, August, September, and November of 1980 and January of 1981. All of the other elements are recorded as unobserved in our data set. Hence, the total number of observation available is 3653.

Option prices have a relatively shorter history. They are observed from 1988-2010 with 265 observation dates. The option contracts expire around 3/4 of a month prior to the expiration of the underlying futures contracts. According to CBOT regulations, the last trading day for an option is the Friday which precedes by at least five business days the last business day of the month preceding the option month. The expiration date is the Saturday following the last trading day. At any particular date, a variety of option contracts, which have different underlying futures contracts and/or strike prices, are traded at the CBOT. The longest maturity date for the option's underlying futures contract is 13 months in our sample. No option contracts are written on futures with more than 13 months until maturity. There could be many options traded on the same futures contracts, but with different strike prices. Our model applies only to European options, while the options traded in the United States are American options. It is well known that American options are more valuable than their European counterparts, because of the existence of early exercise opportunity for American options. However, after calculating values for both American- and European-type options, Plato concluded (1985) that the difference between the two for near-the-money option values is negligible. In our data set, we select option prices with the strike price just above the futures price and the one that just below the strike price to estimate the model. Our ideal option prices data set consists of 265×26 observations for call and put options. However, for the same reason as for futures, many observations are missing,

5.2 Empirical method

Bayesian MCMC methods are employed to estimate the parameter set $\Phi \equiv \{V, \lambda_y, u_x, u_y, k_x, k_y, \rho, \sigma_e, \sigma_c, \sigma_p\}$. The risk-free interest rate is a constant in our two factor model, and is fixed at 5%. We use non-informative priors for u_x, u_y, k_x, k_y, ρ, and for the risk premium parameters λ_y. The prior information for $V, \sigma_e, \sigma_c, \sigma_p$ is proposed as follows,

$$\begin{aligned} V &\sim Inv - Wishart_{\nu^0}(\Lambda^0) \\ \sigma_e &\sim Inv - \chi^2(\nu_e^0, \sigma_e^0) \\ \sigma_c &\sim Inv - \chi^2(\nu_c^0, \sigma_c^0) \\ \sigma_p &\sim Inv - \chi^2(\nu_p^0, \sigma_p^0) \end{aligned}$$

We estimate the parameters by using two different data sets. First, we use futures data only to estimate the parameter set $\{V, \lambda_y, u_x, u_y, k_x, k_y, \rho, \sigma_e\}^1$. Then the whole panel data of both futures prices and option prices across different maturities and exercise prices are employed to estimate $\{V, \lambda_y, u_x, u_y, k_x, k_y, \rho, \sigma_e\}^2$ and σ_c, σ_p. By comparing $\{V, \lambda_y, u_x, u_y, k_x, k_y, \rho, \sigma_e\}^1$ and $\{V, \lambda_y, u_x, u_y, k_x, k_y, \rho, \sigma_e\}^2$, we will be able to assess whether futures and option prices are driven by the same set of underlying parameters. The empirical method used here is designed for the generalized model. The Schwartz model can be easily retrieved by putting the corresponding restrictions into the procedure. The advocated MCMC iteration steps are as follows:

Step 1. Initialization: Specify starting parameter values

$\Phi^{(0)} \equiv \{V^{(0)}, \lambda_y^{(0)}, u_x^{(0)}, u_y^{(0)}, k_x^{(0)}, k_y^{(0)}, \rho^{(0)}, \sigma_e^{(0)}, \sigma_c^{(0)}, \sigma_p^{(0)}\}$. The unobserved factors $\mu_t^{(j)}$ are also estimated by solving equation (17). (Note the perfectly correlated futures prices are selected to be among the observed ones.) The missing observations of futures prices are estimated using equation (12) evaluated at $\Phi^{(j)}$ to get $X_t^{(j)}$, where the M elements of $X_t^{(j)}$ consists of the m ($0 \leq m < M$) missing observation estimates, and $M - m$ observed data points. Similar treatment is applied to the option price, $C_t^{(j)}$, and $P_t^{(j)}$ is filled by evaluating equations (14) and (15) at $\Phi^{(j)}$.

Step 2. Given $[X_t^{(j)}, V^{(j)}, \lambda_y^{(j)}, u_x^{(j)}, \sigma_e^{(j)}, \sigma_c^{(j)}, \sigma_p^{(j)}, \mu_t^{(j)}]$, estimate $[u_y^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \ \rho^{(j+1)}]$ by an effective adaptive, general purpose MCMC Algorithm called T-walk developed by Christen and Fox (2007). The T-walk compares the likelihood of observing futures prices and state variables given (i) and (i$'$) with (ii) and (ii$'$)

(i) $[X_t^{\cdot(j)} - \alpha_t^{\cdot}(\Phi^{(j)}) - \beta_t^{\cdot}(\Phi^{(j)})\mu_t^{(j)}] \sim N(\underline{0}_{((M-2)\times 1)}, \sigma_e^{2(j)}\Omega_{e,t}^{(j)})$

(i$'$) $\begin{bmatrix} x_t^{(j)} - (1-k_x^{(j)})x_{t-1}^{(j)} - \frac{1}{k_x^{(j)}}(u_{x(t)}^{(j)} - (1-k_x^{(j)})u_{x(t-1)}^{(j)}) \\ y_t^{(j)} - (1-k_y^{(j)})y_{t-1}^{(j)} - \frac{1}{k_y^{(j)}}(u_{y(t)}^{(j)} - (1-k_y^{(j)})u_{y(t-1)}^{(j)}) \end{bmatrix} \sim N(\underline{0}_{(2\times 1)}, V^{(j)})$

(ii) $[X_t^{\cdot(j)} - \alpha_t^{\cdot}(\Phi^{(prop)}) - \beta_t^{\cdot}(\Phi^{(prop)})\mu_t^{(prop)}] \sim N(\underline{0}_{((M-2)\times 1)}, \sigma_e^{2(j)}\Omega_{e,t}^{(j)})$

(ii$'$) $\begin{bmatrix} x_t^{(prop)} - (1-k_x^{(prop)})x_{t-1}^{(prop)} - \frac{1}{k_x^{(prop)}}(u_{x(t)}^{(prop)} - (1-k_x^{(prop)})u_{x(t-1)}^{(prop)}) \\ y_t^{(prop)} - (1-k_y^{(prop)})y_{t-1}^{(prop)} - \frac{1}{k_y^{(prop)}}(u_{y(t)}^{(prop)} - (1-k_y^{(prop)})u_{y(t-1)}^{(prop)}) \end{bmatrix} \sim N(\underline{0}_{(2\times 1)}, V^{(j)})$

If option price data are also employed in the estimation process, we also need to compare the relative likelihood of observing the options data given the jth iteration of the estimated parameters and the proposed ones.

(iii) $[C_t^{(j)} - \exp[r(t-T_1)][F(\mu^{(j)}(t), t, T)N(d_1^{(j)}) - KN(d_2^{(j)})]] \sim N(0, \sigma_c^{2(j)})$

(iii′) $[P_t^{(j)} - \exp[r(t-T_1)][F(\mu^{(j)}(t),t,T)N(d_1^{(j)}) - KN(d_2^{(j)})] - \exp[r(t-T_1)][K - F(\mu^{(j)}(t),t,T)]] \sim N(0,\sigma_p^{2(j)})$

(iv) $[C_t^{(prop)} - \exp[r(t-T_1)][F(\mu^{(prop)}(t),t,T)N(d_1^{(prop)}) - KN(d_2^{(prop)})]] \sim N(0,\sigma_c^{2(j)})$

(iv′) $[P_t^{(prop)} - \exp[r(t-T_1)][F(\mu^{(prop)}(t),t,T)N(d_1^{(prop)}) - KN(d_2^{(prop)})] - \exp[r(t-T_1)][K - F(\mu^{(j)}(t),t,T)]] \sim N(0,\sigma_p^{2(j)})$

The unobserved prices in $X_t^{(j)}$ and $\mu_t^{(j)}$are then updated after obtaining $[u_y^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}]$.

Step 3. Given $[X_t^{(j)}, \lambda_y^{(j)}, u_x^{(j)}, \sigma_e^{(j)}, \sigma_c^{(j)}, \sigma_p^{(j)}, \mu_t^{(j+1)}, u_y^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}]$, use Metropolis Hastings algorithm to generate $V^{(j+1)}$.

Generate $V^{(prop)} \sim Inv - Wishart_{Nobs-3}((Nobs-3)V^{(j)})$.

Calculate the acceptance ratio, $R = \frac{P(V^{(prop)}|data)/P(V^{(prop)}/V^{(j)})}{P(V^{(j)}|data)/P(V^{(j)}/V^{(j=prop)})}$

Generate a random variable γ from a standard uniform distribution.

If $\gamma < R$, $V^{(j+1)} = V^{(prop)}$, otherwise $V^{(j+1)} = V^{(j)}$. $X_t^{(j)}$ and $\mu_t^{(j)}$ are then updated after obtaining $V^{(j+1)}$.

Step 4. Given $[\sigma_e^{(j)}, \sigma_c^{(j)}, \sigma_p^{(j)}, \mu_t^{(j)}, u_{y,}^{(j+1)}\ k_x^{(j+1)}, k_y^{(j+1)}\rho^{(j+1)}, V^{(j+1)}]$, draw $\lambda_y^{(j+1)}$ and $u_x^{(j+1)}$. Recall equation (16), $\lambda_y^{(j+1)}$ and $u_x^{(j+1)}$ can be generated by a GLS regression.

Step 5. Given $[\sigma_c^{(j)}, \sigma_p^{(j)}, \mu_t^{(j)}, u_{y,}^{(j+1)}\ k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}, V^{(j+1)}, \lambda_y^{(j+1)}, u_x^{(j+1)}]$, estimate $\sigma_e^{(j+1)}$.

$\sigma_e^{(j+1)}$ | future data $\sim Inv - \chi^2(\nu_e^0 + n, \frac{\nu_e^0\sigma_e^0 + n\nu_e}{\nu_e^0 + n})$

Steps 6 and 7 are necessary only if the option data set is also used in estimating Φ.

Step 6. Given $[\sigma_p^{(j)}, \mu_t^{(j)}, u_{y,}^{(j+1)}\ k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}, V^{(j+1)}, \lambda_y^{(j+1)}, u_x^{(j+1)}, \sigma_e^{(j+1)}]$, generate $\sigma_c^{(j+1)}$.

$\sigma_c^{(j+1)}$ | call option data $\sim Inv - \chi^2(\nu_c^0 + n, \frac{\nu_c^0\sigma_c^0 + n\nu_c}{\nu_c^0 + n})$

Step 7. Given $[\mu_t^{(j)}, u_{y,}^{(j+1)}\ k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}, V^{(j+1)}, \lambda_y^{(j+1)}, u_x^{(j+1)}, \sigma_e^{(j+1)}, \sigma_c^{(j+1)}]$, generate $\sigma_p^{(j+1)}$.

$\sigma_p^{(j+1)} \mid$ put option data $\sim Inv - \chi^2(\nu_p^0 + n, \frac{\nu_p^0 \sigma_p^0 + n\nu_p}{\nu_p^0 + n})$

Step 8. Update the unobserved prices to get $X_t^{(j+1)}, C_t^{(j+1)}, \lambda_y^{(j+1)}$ and $\mu_t^{(j+1)}$ given $[u_{y,}^{(j+1)}\ k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}, V^{(j+1)}, \lambda_y^{(j+1)}, u_x^{(j+1)}, \sigma_e^{(j+1)}, \sigma_c^{(j+1)}, \sigma_p^{(j+1)}]$.

Step 9. Set $j = j + 1$.

Step 10. If the maximum iteration is reached, stop. Otherwise, go to Step 2.

5.3 Estimation results

The parameter estimates obtained from the Bayesian MCMC estimation approach for the Schwartz model and the generalized model are given in table 1a for futures data only and table 1b for both futures prices and options prices.

[INSERT TABLE 1a ABOUT HERE]

[INSERT TABLE 1b ABOUT HERE]

Comparing Table 1a with Table 1b, we can see that the estimated parameters from the two data sets are very similar. There is large overlapping between the 95% estimated posterior intervals. The parameter estimates using only futures data are very similar to the estimates using both futures and option data. We can not reject that $\{V, \lambda_y, K_0, K_1, \rho, \sigma_e\}^1$ is equal to $\{V, \lambda_y, K_0, K_1, \rho, \sigma_e\}^2$. k_x is significantly positive, which provides empirical support for the postulation that the process of spot prices is mean reverting. Parameter u_x in Schwartz's model describes the expected appreciation rate of the non-stationary state variable (log of the spot price). The non-seasonality part $u_{x,0}$ is not significantly different from zero. The estimates in Table 1a indicates that the mean-reversion parameter k_c, and k_y are positive and, hence, that the state variable c in the Schwartz model and and y in our model are indeed stationary in the soybean market. The median of the estimated k_c, and k_y is about 0.0901 and 0.0933, corresponding to half-lives of 3.34 months and 3.23 months, respectively. [2] Similarly, the half lives of the stationary factor x_t in our model is about five years. Both the Schwartz model and our model report a similar volatility of the two Gaussian factors. Moreover, the instantaneous correlation coefficient, ρ, between the two factors is significantly positive (0.7037) in the Schwartz's model. However, it turns out to be not significantly different from zero, when we separate the correlated part from the convenience yield. The non-seasonal part of risk premia associated with the convenience yield process is not significantly different from zero in both models.

[2] Note, the half-life expresses the time it takes before a given shock to this process is expected to have leveled off by half of the shock; the half-life in the Ornstein-Uhlenbeck process is calculated as $\log(2)/k$.

The estimated seasonal components of u_x in the generalized model are described by the estimates of $u_{x,1,\sin}, u_{x,1,\cos}, u_{x,2,\sin}$, and $u_{x,2,\cos}$. As apparent from the estimation results in Table 1a, the seasonal component is a very important feature with respect to the model's ability to fit the soybean futures prices data, since 3/4 of the seasonal parameters are significant at the 95% level.

Figure 1 shows the term structure of soybean futures price for January 15 2010, the last date of our observation. We show the term structure implied by both models.

[INSERT FIGURE 1 ABOUT HERE]

The X-axis is the time to maturity of the futures contract. It is projected with times to maturity up to 60 months. The Y-axis is the corresponding futures prices measured in cents per bushel. When time to maturity runs over 15 months, Schwartz projections exceed our projections. The black, blue, and brown lines represent the 97.5, 50 and 2.5 percentile of the estimation by the models. We also plot the observed real world data on the same picture which is represented by the green points. The red points are also observed from the market. However, their trading volume was zero. If a contract has not traded on a day, the settlement price we observed represents a "best guess" by the clearing house of where it would have traded had it traded at the settlement time. We can see that both models fit the sample data pretty well. These two models tend to agree with each other in pricing futures with maturities of less than 2 years. However, for futures contracts with longer maturities there is a consistent discrepancy between the two models. The generalized model predicts consistently lower price than the Schwartz' model. On January 2010, the spot price for soybean was relatively high in history. So the projection of futures prices by our model is downward trending in the long term, since mean-reverting is assumed in our model. On the other hand, Schwartz predicts that the spot price will continue growing at the estimated appreciation rate due to the basic assumption of Geometric Brownian motion in the spot price. The projections exhibit seasonality. The models excellently fit the seasonality of the observed futures price (green points). It can be seen that futures price reaches a local minimum when the expiration month is right after the harvest season (November). They reach a local maximum around July, which is consistent with our observation of historical futures prices.

Table 2 provides a cross-section comparison between the Schwartz model and the generalized model. It reports the root mean squared error (RMSE) and mean error in monetary terms for both the Schwartz and the generalized model prediction on futures prices. The results are grouped according to contracts with different times to maturity.

[INSERT TABLE 2 ABOUT HERE]

This information should give us an indication of the relative performance of the models. Comparing RMSE, it is obvious that our model reports a consistently lower value of RMSE for all the futures contracts in the group, which is an indicator of better fitting the data set. In terms of mean error, our model reports a smaller (in absolute value) mean error for futures with shorter maturities, while Schwartz's model has smaller error for futures with longer maturities.

The estimates of σ_e, σ_c, and σ_p describe the inferred standard deviation on the noise terms that allow for some deviation between theoretical and observed (log) futures prices, call option prices and put option prices, respectively. σ_e is smaller in the generalized model but not significant, it is about 2.55% of the soybean futures prices. The median estimates of σ_c and σ_p are also smaller in the generalized model, however they are not significantly lower either. They represents about 30% of the call option price and 35 % of the put option price. The main source of this noise in our specific data set may be that the settlement prices are set by the administrators at CBOT, thus they may not exactly match the market price. Errors in data registration, price limit and handling of bid-ask spread may also contribute to the "noise term".

Figure 2 shows the term structure of the call option price. The X-axis is the time to maturity of the underlying futures contract. The Y-axis is the premium of an at-the-money call option. The black line is predicted by the Schwartz model, while the red line is predicted by the generalized model. These lines represent the median of the projection by the two models. Similar to the futures price projection, it is observed that the prediction from the two models are very close when time to maturity of the underlying future contracts is less than 2 years. However, as the length of maturity increases, the difference is enlarged. This discrepancy is consistent with the key assumption in the generalized model that there is mean-reversion in the spot price. With the restriction that the spot price level will revert, one can predict that the price level will revert to the expected production cost rather than trend as predicted by the Schwartz model. This additional piece of information allows us to reduce the futures volatility level, which will in turn reduce the price of the options.

[INSERT FIGURE 2 ABOUT HERE]

We use observed futures prices to predict the at-the-money call prices with maturities less than or equal to 13 months. For maturities from 14 to 100 months, expected prices by Schwartz's futures model (generalized model) are employed to project option prices using the option pricing formula derived from Schwartz' model (generalized model). When time to maturity is less than 14 month, the Schwartz option formula and the generalized option formula face the same underlying futures prices and strike prices. So, it is reasonable to examine the difference of the projections between the Schwartz model and our model. The difference is summarized in figure 3 for call options.

[INSERT FIGURE 3 HERE]

We can see that with time maturity less than 14 months these two models predict very close values for both at-the-money call and put options. The difference is within two cents. One may argue that since the Schwartz model predicts consistently higher futures price for longer maturity, and the at-the-money option price is increasing with the underlying futures price, that may be responsible for the enlarged gap between these two models' projection for longer maturity. This fact does contribute to the gap in figure 2. However, even if we normalized the option's price using the underlying futures price, the Schwartz model still anticipates a higher option price. The normalized at the money options prices are shown in figure 4.

[INSERT FIGURE 4 HERE]

6 Conclusion

We generalize Schwartz's two-factor model by allowing mean reversion in spot prices, which is a key feature in agricultural commodity markets. Closed form futures-pricing formulas are derived based on our model. We show that the Schwartz model is a special case of our model. Agricultural commodity markets exhibit clear seasonality patterns. We introduce seasonality into our model by adding periodical functions to the parameters associated with commodity prices.

Real-world futures and option prices observed from the CBOT are employed to estimate the models. A Bayesian MCMC algorithm is developed to estimate our model. Estimates for the Schwartz model are obtained by imposing the corresponding restrictions to our model. In January 2010, the spot prices of soybean were historically high. The basic assumption of the Schwartz model tell us that the prices will continue growing at the expected appreciation rate. However, our model suggest that prices will gradually return to the long run mean. By comparing the two models' predictions of the term structure of futures (on January 15 2010), we can conclude that for futures contract with short term maturity (less than 15 months), the two models' predictions are very similar. However, for longer maturities, our model consistently reports lower futures prices than Schwartz's model. The cross-section comparison suggests that our model better fits the data set.

Finding significant mean reversion in commodity spot prices has important practical implications on commodity options. The return forecastability associated with mean reversion in spot prices implies that return variances do not increase linearly with the measurement interval, as they would if prices followed a random walk. Our model admits mean-reverting that consistently lower volatility on the future prices, which will in turn reduce the option prices written on the futures contracts, especially ones with a long time to maturity.

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Appendix A

$$\begin{aligned}\beta^{'}(t) &= K_1^T\beta(t)\\ \alpha^{'}(t) &= -K_0(t)\cdot\beta(t)-\frac{1}{2}\beta^T(t)V\beta(t)\end{aligned}$$

with boundary conditions $\beta(T)=\phi,\ \ \alpha(T)=0.$

$$\begin{aligned}\beta_1^{'}(t) &= k_x\beta_1(t), \text{together with boundary conditions} \Rightarrow \beta_1(t)=e^{k_x(t-T)}\\ \beta_2^{'}(t) &= \beta_1(t)+k_y\beta_2(t)\\ &\Rightarrow \beta_2^{'}(t)=e^{k_x(t-T)}+k_y\beta_2(t)\\ &\Rightarrow \beta_2(t)=\frac{\exp(k_x(t-T))-\exp(k_y(t-T))}{k_x-k_y}\end{aligned}$$

$$\begin{aligned}\alpha^{'}(t) &= -K_0(t)\cdot\beta(t)-\frac{1}{2}\beta^T(t)V\beta(t)\\ &= r\beta_1(t)+\theta(t)\beta_2(t)-\frac{1}{2}\beta^T(t)V\beta(t)\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\alpha(t) &= \frac{r}{k_x}(e^{k_x(t-T)}-1)+\frac{\theta_0}{k_x-k_y}[(\frac{1}{k_x}(\exp(k_x(t-T))-1)-(\frac{1}{k_y}(\exp(k_y(t-T))-1)]\\ &+\sum_{h=1}^{2}\frac{\theta_{h,\cos}}{k_x-k_y}(\frac{s^2}{k_x^2s^2+4\pi^2h^2}-\frac{s^2}{k_y^2s^2+4\pi^2h^2})\times\\ &\{k_x[\exp(k_x(t-T))\cos(\frac{2\pi h}{s}t)-\cos(\frac{2\pi h}{s}T)]+\frac{2\pi h}{s}[\exp(k_x(t-T))\sin(\frac{2\pi h}{s}t)-\sin(\frac{2\pi h}{s}T)]\}\\ &+\sum_{h=1}^{2}\frac{\theta_{h,\sin}}{k_x-k_y}(\frac{s^2}{k_x^2s^2+4\pi^2h^2}-\frac{s^2}{k_y^2s^2+4\pi^2h^2})\times\\ &\{k_x[\exp(k_x(t-T))\sin(\frac{2\pi h}{s}t)-\sin(\frac{2\pi h}{s}T)]+\frac{2\pi h}{s}[\exp(k_x(t-T))\cos(\frac{2\pi h}{s}t)-\cos(\frac{2\pi h}{s}T)]\}\\ &-\frac{1}{2}\{[\frac{\sigma_x^2}{2k_x}+\frac{\rho_{xy}\sigma_x\sigma_y}{k_x(k_x-k_y)}+\frac{\sigma_y^2}{2k_x(k_x-k_y)^2}]\times[\exp(2k_x(t-T))-1]\\ &-[\frac{2\rho_{xy}\sigma_x\sigma_y}{k_x^2-k_y^2}+\frac{2\sigma_y^2}{(k_x+k_y)(k_x-k_y)^2}]\times[\exp((k_x+k_y)(t-T))-1]\\ &+\frac{\sigma_y^2}{2k_y(k_x-k_y)^2}[\exp(2k_y(t-T))-1]\}\end{aligned}$$

Define

$$\begin{aligned}
\eta_1(t) \quad \equiv \quad & \frac{r}{k_x}(e^{k_x(t-T)}-1)+\frac{\theta_0}{k_x-k_y}[(\frac{1}{k_x}(\exp(k_x(t-T))-1)-(\frac{1}{k_y}(\exp(k_y(t-T))-1)] \\
& +\sum_{h=1}^{2}\frac{\theta_{h,\cos}}{k_x-k_y}(\frac{s^2}{k_x^2s^2+4\pi^2h^2}-\frac{s^2}{k_y^2s^2+4\pi^2h^2})\times \\
& \{k_x[\exp(k_x(t-T))\cos(\frac{2\pi h}{s}t)-\cos(\frac{2\pi h}{s}T)]+\frac{2\pi h}{s}[\exp(k_x(t-T))\sin(\frac{2\pi h}{s}t)-\sin(\frac{2\pi h}{s}T)]\} \\
& +\sum_{h=1}^{2}\frac{\theta_{h,\sin}}{k_x-k_y}(\frac{s^2}{k_x^2s^2+4\pi^2h^2}-\frac{s^2}{k_y^2s^2+4\pi^2h^2})\times \\
& \{k_x[\exp(k_x(t-T))\sin(\frac{2\pi h}{s}t)-\sin(\frac{2\pi h}{s}T)]+\frac{2\pi h}{s}[\exp(k_x(t-T))\cos(\frac{2\pi h}{s}t)-\cos(\frac{2\pi h}{s}T)]\}
\end{aligned}$$

$$\begin{aligned}
\eta_2(t) \equiv \qquad & -\frac{1}{2}\{[\frac{\sigma_x^2}{2k_x}+\frac{\rho_{xy}\sigma_x\sigma_y}{k_x(k_x-k_y)}+\frac{\sigma_y^2}{2k_x(k_x-k_y)^2}]\times[\exp(2k_x(t-T))-1] \\
& -[\frac{2\rho_{xy}\sigma_x\sigma_y}{k_x^2-k_y^2}+\frac{2\sigma_y^2}{(k_x+k_y)(k_x-k_y)^2}]\times[\exp((k_x+k_y)(t-T))-1] \\
& +\frac{\sigma_y^2}{2k_y(k_x-k_y)^2}[\exp(2k_y(t-T))-1]\}
\end{aligned}$$

Then $\alpha(t)=\eta_1(t)+\eta_2(t)$

Appendix B

The call option formula can be obtained by noting that if $\ln[F(\mu(T_1), T_1, T)]$ is distributed as a normal random variable with mean $\varpi(\mu(t), t, T_1, T)$ and variance $\sigma(t, T_1, T)^2$, then

$$\begin{aligned}
E_t^*\{\max[F_{T_1} - K, 0]\} &= \int_{\ln(K)}^{\infty} (F_{T_1} - K)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1}) \\
&= \int_{\ln(K)}^{\infty} F_{T_1}\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1}) \\
&\quad -K\int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1})
\end{aligned}$$

because $F_{T_1} > K \Rightarrow \ln(F_{T_1}) > \ln(K)$. But $\ln(F_{T_1}) > \ln(K) \Rightarrow [\ln(F_{T_1}) - \varpi]/\sigma > [\ln(K) - \varpi]/\sigma = [\ln(K) - \varpi - \sigma^2/2 + \sigma^2/2]/\sigma$.

By equation (13) , $[\ln(K) - \varpi - \sigma^2/2 + \sigma^2/2]/\sigma = [\ln(K) - \ln(F_t) + \sigma^2/2]/\sigma$, so that

$$\begin{aligned}
\int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1}) &= 1 - N\{[\ln(K/F_t) + \sigma^2/2]/\sigma\} \\
&= N\{[\ln(F_t/K) - \sigma^2/2]/\sigma\} \qquad (19)
\end{aligned}$$

In addition,

$$\begin{aligned}
&\int_{\ln(K)}^{\infty} F_{T_1}\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1}) \\
= &\int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\ln(F_{T_1}) - \frac{1}{2}\left[\frac{\ln(F_{T_1}) - \varpi}{\sigma}\right]^2\right\} d\ln(F_{T_1})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\varpi + \frac{\sigma^2}{2} - \frac{1}{2}\left[\frac{\ln(F_{T_1}) - (\varpi + \sigma^2)}{\sigma}\right]^2\right\} d\ln(F_{T_1}) \\
&= \exp(\varpi + \frac{\sigma^2}{2}) \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - (\varpi + \sigma^2/2) - \sigma^2/2}{\sigma}\right]^2\right\} d\ln(F_{T_1}) \\
&= F_t \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(F_{T_1}) - \ln(F_t) - \sigma^2/2}{\sigma}\right]^2\right\} d\ln(F_{T_1}) \\
&= F_t\{1 - N[(\ln(K/F_t) - \sigma^2/2)/\sigma]\} \\
&= F_t N[(\ln(F_t/K) + \sigma^2/2)/\sigma] \qquad (20)
\end{aligned}
$$

So, combine equation (19) and equation (20), we get

$$
E_t^*\{\max[F_{T_1} - K, 0]\} = F_t N[(\ln(F_t/K) + \sigma^2/2)/\sigma] - KN\{[\ln(F_t/K) - \sigma^2/2]/\sigma\}
$$

Table 1a Parameter estimates using futures data only

Parameters	Schwartz Model	Parameters	Advocated Model
k_x	restricted to 0	k_x	$(.0033, .0050, .0061)$
$u_{x,0}$	$(-.0061, .0013, .0087)$	$u_{x,0}$	$(.0196, .0299, .0411)$
k_c	$(.0825, .0901, .0974)$	k_y	$(.0843, .0933, .0992)$
$u_{c,0}$	$(.0001, .0008, .0014)$	$u_{y,0}$	$(-.0035, -.0023, -.0013)$
$\lambda_{c,0}$	$(-.0004, -.0002, .0009)$	$\lambda_{y,0}$	$(-.0008, -.0001, .0005)$
$\lambda_{c,1,\sin}$	$(-.0023, -.0014, -.0005)$	$\lambda_{y,1,\sin}$	$(-.0025, -.0015, -.0006)$
$\lambda_{c,1,\cos}$	$(-.0009, -.0000, .0009)$	$\lambda_{y,1,\cos}$	$(-.0009, -, 0000..0009)$
$\lambda_{c,2,\sin}$	$(.008, .0019, .0030)$	$\lambda_{y,2,\sin}$	$(.0007, .0018, .0029)$
$\lambda_{c,2,\cos}$	$(-.0030, . - 0020, -.0009)$	$\lambda_{y,2,\cos}$	$(-.0033, -.0021, -.0010)$
$u_{x,1,\sin}$	$(.0056, .0161, .0266)$	$u_{x,1,\sin}$	$(.0057, .0162, .0267)$
$u_{x,1,\cos}$	$(-.0022, .0083, .0188)$	$u_{x,1,\cos}$	$(-.0024, -.0081, .0185)$
$u_{x,2,\sin}$	$(-.0196, -.0090, .0016)$	$u_{x,2,\sin}$	$(-.0190, -.0085, .0021)$
$u_{x,2,\cos}$	$(-.0141, -.0035, .0071)$	$u_{x,2,\cos}$	$(-.0138, -.0032, .0074)$
$u_{c,1,\sin}$	$(-.0020, -.0010, -.0001)$	$u_{y,1,\sin}$	$(-.0022, -.0011, -.0002)$
$u_{c,1,\cos}$	$(-.0050, -.0040, -.0031)$	$u_{y,1,\cos}$	$(-.0052, -.0042, -.0032)$
$u_{c,2,\sin}$	$(.0061, .0078, .0094)$	$u_{y,2,\sin}$	$(.0063, .0079, .0096)$
$u_{c,2,\cos}$	$(-.0030, -.0013, -.0003)$	$u_{y,2,\cos}$	$(-.0035, -.0018, -.0002)$
σ_x^2	$(.0710, .0740, .0772)$	σ_x^2	$(.0711, .0740, .0773)$
σ_c^2	$(.0058, .0062, .0066)$	σ_y^2	$(.0058, .0062, .0066)$
$\rho_{x,c}$	$(.6725, .7037, .7325)$	$\rho_{x,y}$	$(-.1865, -.0230, .1672)$
σ_e^2	$(.00063, .00067, .00071)$	σ_e^2	$(.00061, .00065, .00069)$
ρ	$(.0807, .1125, .1444)$	ρ	$(.0842, .1148, .1455)$

Note: The three quantities within parenthesis denote respectively the 2.5, 50 and 97.5 percentile of the posterior probability band.

Table 1b Parameter estimates using the whole data set

Parameters	Schwartz Model	Parameters	Advocated Model
k_x	restricted to 0	k_x	$(.0032, .0051, .0064)$
$u_{x,0}$	$(-.0059, .0013, .0085)$	$u_{x,0}$	$(.0212, .0339, .0450)$
k_c	$(.0753, .0831, .0919)$	k_y	$(.0757, .0840, .0945)$
$u_{c,0}$	$(.0001, .0007, .0013)$	$u_{y,0}$	$(-.0037, -.0026, -.0012)$
$\lambda_{c,0}$	$(-.0004, .0002, .0008)$	$\lambda_{y,0}$	$(-.0009, -.0002, .0004)$
$\lambda_{c,1,\sin}$	$(-.0009, -.0000, .0008)$	$\lambda_{y,1,\sin}$	$(-.0024, -.0015, -.0006)$
$\lambda_{c,1,\cos}$	$(-.0009, -.0001, .0007)$	$\lambda_{y,1,\cos}$	$(-.0009, -.0001, .0008)$
$\lambda_{c,2,\sin}$	$(.0007, .0019, .0030)$	$\lambda_{y,2,\sin}$	$(.0007, .0018, .0029)$
$\lambda_{c,2,\cos}$	$(-.0031, -.0020, . - 009)$	$\lambda_{y,2,\cos}$	$(-.0033, -.0022, -.0011)$
$u_{x,1,\sin}$	$(.0058, .0160, .0262)$	$u_{x,1,\sin}$	$(.0062, .0161, .0260)$
$u_{x,1,\cos}$	$(-.0019, .0083, .0185)$	$u_{x,1,\cos}$	$(-.0018, .0081, .0179)$
$u_{x,2,\sin}$	$(-.0193, -.0089, .0014)$	$u_{x,2,\sin}$	$(-.0183, -.0084, .0016)$
$u_{x,2,\cos}$	$(-.0137, -.0035, .0069)$	$u_{x,2,\cos}$	$(-.0130, -.0030, -.0070)$
$u_{c,1,\sin}$	$(-.0019, -.0010, -.0001)$	$u_{y,1,\sin}$	$(-.0020, -.0011, -.0002)$
$u_{c,1,\cos}$	$(-.0049, -.0040, -.0031)$	$u_{y,1,\cos}$	$(-.0051, -.0042, -.0032)$
$u_{c,2,\sin}$	$(.0059, .0077, .0095)$	$u_{y,2,\sin}$	$(.0062, .0080, .0098)$
$u_{c,2,\cos}$	$(-.0031, -.0013, -.0003)$	$u_{y,2,\cos}$	$(-.0036, -.0019, -.0003)$
σ_x^2	$(.0681, .0721, .0757)$	σ_x^2	$(.0663, .0698, .0734)$
σ_c^2	$(.0055, .0059, .0065)$	σ_y^2	$(.0053, .0058, .0063)$
$\rho_{x,c}$	$(.6437, .6970, .7421)$	$\rho_{x,y}$	$(-.1296, .0363, .2307)$
σ_e^2	$(.00063, .00067, .00071)$	σ_e^2	$(.00061, .00065, .00069)$
σ_c^2	$(.0951, .1048, .1169)$	σ_c^2	$(.0869, .0941, .1034)$
σ_p^2	$(.1211, .1351, .1514)$	σ_p^2	$(.1048, .1172, .1219)$
ρ	$(.0740, .1115, .1482)$	ρ	$(.0782, .1122, .1449)$

Note: The three quantities within parenthesis denote respectively the 2.5, 50 and 97.5 percentile of the posterior probability band.

Table 2 Cross-Section Comparison Between Schwartz' Model and Generalized Model

	RMSE		Mean Error	
Model Contract	Schwartz	Generalized	Schwartz	Generalized
F2	0.0066	0.0061	-0.0147	-0.0146
F3	0.0187	0.0179	0.0050	0.0043
F4	0.0215	0.0207	0.0051	0.0048
F5	0.0276	0.0256	0.0165	0.0162
F6	0.0278	0.0258	0.0128	0.0129
F7	0.0295	0.0269	0.0232	0.0246
F8	0.0274	0.0251	0.0186	0.0231
F9	0.0263	0.0241	0.0238	0.0293
F10	0.0239	0.0224	0.0173	0.0243
F11	0.0153	0.0144	0.0189	0.0268
F12	0.0086	0.0081	0.0056	0.0168

Note: Price is measured in dollars.

FN is the futures contract that matures in N months.

Figure 1: Projection of futures price by the Schwartz model and the advocated model

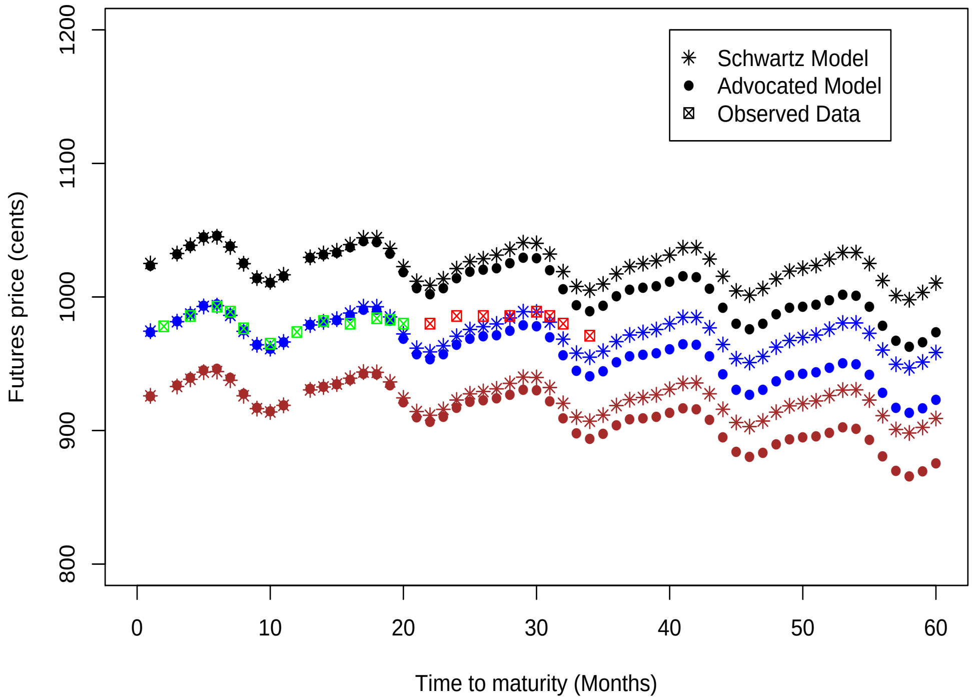

Figure 2: Projection of at the money call option price

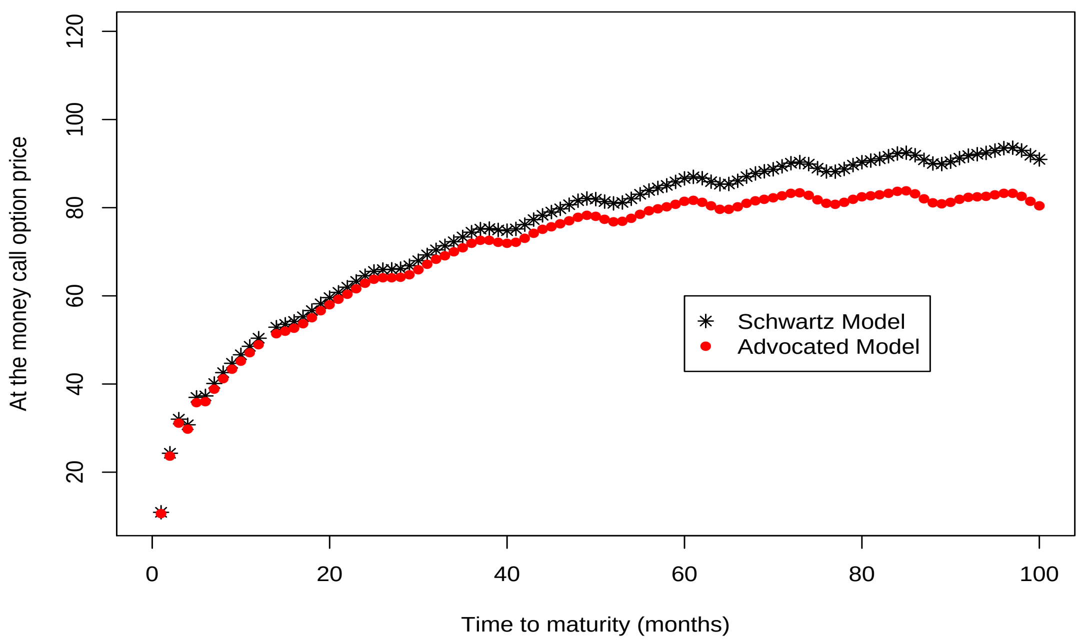

Figure 3: Projection of the within-sample difference of call option prices

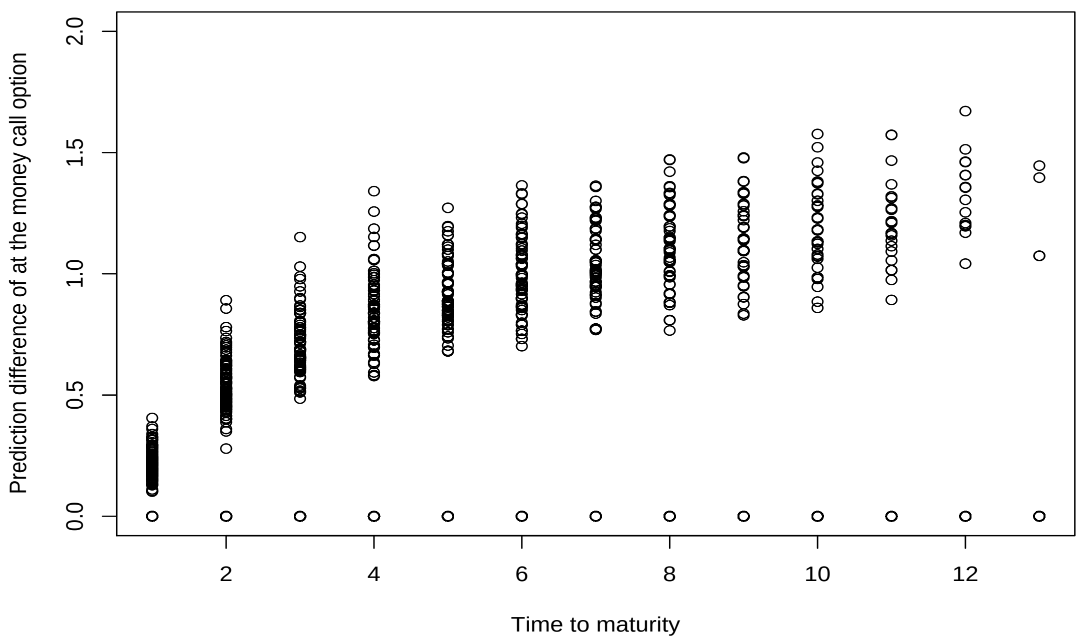

Figure 4: Projection of the normalized call option price