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Optimal Hedging with Views: a Bayesian Approach

Practitioners Abstract

The optimal hedging model has become the standard theoretical model of normative hedging behavior due to its intuitive tradeoff of expected return with risk, its efficient use of information and its easy implementation. In practice, the model can be easily implemented with the Parameter Certainty Equivalent procedure, which substitutes sample estimates for the true but unknown model parameters. But subjective views, which refer to opinions concerning the directions of market returns of the assets involved in hedging decisions, are either completely ignored or handled in an ad hoc and unsatisfactory manner within the optimal hedging model. Given the widespread use of subjective views in hedging practice and the potential economic benefit of selective hedging, the lack of accommodation of subjective views in the optimal hedging model is a serious problem and could hamper the model's application in risk management practice. With an empirical Bayesian approach adopted, this study proposes an alternative Bayesian optimal hedging model, in which a hedger can adjust his/her optimal hedging position (ratio) according to his/her view(s) on the expected returns of assets under consideration. Like Lence and Hayes' Bayesian optimal hedging model (1994a, 1994b), the optimal hedging position is also determined by mean-variance optimization conditioned on the predictive expectation vector and predictive covariance matrix of asset returns, but unlike their model, the number and type of subjective views that can be expressed is quite flexible because of the adoption of an empirical Bayesian approach. The empirical Bayesian optimal hedging model provides practitioners with a theoretically intuitive yet quantitatively rigorous framework to blend subjective views and the market consensus estimated from sample data according to their relative confidence levels.

Keywords: optimal hedging, parameter estimation risk, subjective view(s), empirical Bayesian approach

1 Introduction

The optimal hedging model has become the standard theoretical model of normative hedging behavior due to its intuitive tradeoff of expected return with risk, its efficient use of information and its easy implementation. In empirical applications, the model has often been implemented with the Parameter Certainty Equivalent (PCE) procedure, which substitutes sample estimates for the model's parameters in determining the optimal hedging position (ratio). However, despite its intuitive appeal and easy implementation, the PCE procedure has been criticized for its complete ignorance of parameter estimation risk (Lence and Hayes 1994a; Lence and Hayes 1994b). The PCE procedure simply substitutes sample estimates for the true but unknown parameters, completely ignoring estimation errors in parameters which often results in sub-optimal hedging positions (ratios). Lence and Hayes (1994a, 1994b), based on a Bayesian portfolio optimization framework, develop a Bayesian optimal hedging model to effectively tackle parameter estimation risk. However within the optimal hedging context, subjective views, which refer to opinions concerning the direction of market returns of the assets involved in hedging decisions, are still either completely ignored or handled in an ad hoc and unsatisfactory manner.

It has long been recognized that hedging with a view to market direction is commonplace in agricultural markets. Working (1962, p.440) observed many years ago that large and well managed firms in the grain trade sometimes hedge incompletely or they hedge nothing, and

that firms hedge selectively according their price expectations. Furthermore he argued that “Outside the grain and cotton trades, . . . , selectivity in hedging is so common as to suggest that in a considerable number of futures markets the greater part of short hedging done may be selective.” More recent evidence can be found in studies of the behavior of agricultural market advisory services (Bertoli et al. 1999; Martines-Filho et al. 2003a; Martines-Filho et al. 2003b). These studies find the hedging positions recommended by advisory services to crop producers often display large cross-sectional and time-series variation, a result that is difficult to explain without recourse to widely varying subjective views. Finally, using data from a real-time forward pricing game employed by Maryland grain marketing clubs from 1994 through 1998, McNew and Musser (2002) find that “farmers may form different expectations than those conveyed by forward prices.”

Surveys show that financial practitioners also often have subjective views on market trends and sometimes hedge selectively based on their views. For example, Bondnar, Hayt and Marston (1996, 1998) find that over a third of U.S. non-financial firms that use derivatives in risk management said that they sometimes actively took positions based on their views on markets. Dolde (1993) surveys the derivatives use of the Fortune 500 companies, finding that almost 90 percentage of derivative users said that they sometimes took views on markets. With a sample of quarterly hedging positions of 48 firms from three industries, including 44 gold producers, Brown, Crabb and Haushalter (2002) find that many managers incorporate their market views into risk management policies, and therefore, hedge selectively.

Although empirical studies provide mixed evidence on economic gains from selective hedging (Brown, Crabb, and Haushalter 2002; Adam and Fernando 2003; Irwin, Martines-Filho, and Good 2003), selective hedging based on subjective views could still be a value-adding form of risk-taking for practitioners who have a comparative advantage in information gathering and processing. Stultz (1996) suggests that some companies may acquire specialized information about certain markets through their normal operating activities, and therefore, may benefit from hedging selectively according to their subjective views.

Despite the widespread use of subjective views in hedging practice and the potential economic benefit of selective hedging, the optimal hedging model cannot consistently and effectively accommodate subjective views. The lack of accommodation of subjective views in the optimal hedging model could hamper its application in practice. The goal of this study is not to take a stand on the merits of subjective views, instead it is to model how practitioners should hedge given their subjective views on market returns.

The subjective view problem, within the optimal hedging context, should be tackled together with the parameter estimation risk problem given the inherent link between them. The type and magnitude of the subjective views that a practitioner can have is closely related with the perceived estimation errors in parameters. Specifically, if practitioners think that they can estimate the model’s parameters quite precisely, it is less likely they will have “views” on the model’s parameters concerning asset returns. However given a whole array of different econometric estimation methods and possibly different sample data, it is quite likely that practitioners will obtain parameter estimates that are different than “market consensus” estimates or “firm-wide forecasting model” estimates, which, respectively, represent market-wide or firm-wide consensus on these parameters.

The Bayesian portfolio optimization framework provides a theoretically intuitive and quantitatively rigorous framework to consistently and effectively accommodate both subjective views and parameter estimation risk (Brown 1979; Jorion 1985; Frost and Savarino 1986; Jorion 1986; Black and Litterman 1992; Pastor 2000). The Bayesian model determines the optimal portfolio weight via expected utility maximization (mean-variance optimiza-

tion) conditioned on the predictive distribution (predictive expectations and covariance matrix) of asset returns, which explicitly incorporates subjective views and parameter estimation risk through prior specification and Bayesian updating.

With the Bayesian portfolio optimization framework applied to optimal hedging, Lence and Hayes (1994a, 1994b) develop a Bayesian optimal hedging model. Their model can effectively tackle the parameter estimation risk problem, but can only accommodate subjective views under some restrictive and unrealistic assumptions. Lence and Hayes (1994b, p.356) claim “in general, the decision maker will have relevant prior and sample information, in this instance, the model advanced can be used to optimally blend both types of information.” However their model requires practitioners to calibrate the prior distribution with non-sample information, which includes subjective views. This requirement is likely to be unrealistic for many practitioners, who may not have enough information to calibrate the entire prior distribution. The subjective views of practitioners may relate only to one or two parameters of the prior distribution, most particularly the expected returns, or depict the relative relation of the parameters of the prior distribution.

Instead, an empirical Bayesian approach may be adopted to tackle subjective views and parameter estimation risk. Black and Litterman (1990, 1992) adopt an empirical Bayesian approach to accommodate both investors’ subjective views and parameter estimation risk within the standard mean-variance portfolio optimization model. With the empirical Bayesian approach, the prior distribution is calibrated with sample data, which should contain enough information for estimating all parameters of the prior distribution. Subjective views, treated as new information, are used to update the prior. The number and type of subjective views that can be expressed is quite flexible; investors can have one or more absolute or relative views concerning expected asset returns. With the market portfolio as a neutral reference point, investors can blend their subjective views with the market equilibrium and invest in a manner consistent with their subjective views. Because of its theoretical intuition and easy implementation, the Black-Litterman model has been gaining wide acceptance and application in portfolio management practice (Bevan and Winkelmann 1998; He and Litterman 1999).

However, the Black-Litterman model is not completely consistent with the aforementioned Bayesian portfolio optimization framework, which requires both the predictive expectation vector and predictive covariance matrix of asset returns to be used as inputs in the mean-variance portfolio optimization. Instead, the Black-Litterman model, when computing the optimal portfolio, still uses the sample covariance matrix of asset returns in place of the predictive covariance matrix. Even if subjective views have only been expressed on expected returns, the impact of the subjective views should flow into the covariance matrix of asset returns through the Bayesian updating process.

Given the portfolio interpretation of optimal hedging, we also adopt an empirical Bayesian approach in this study to accommodate both subjective views and parameter estimation risk within an optimal hedging context. Like Lence and Hayes’ Bayesian model, the model also determines optimal hedging ratios via mean-variance optimization conditioned on the predictive expectation vector and predictive covariance matrix of asset returns. Unlike their model, because of the empirical Bayesian approach, the new Bayesian model gives practitioners flexibility in specifying subjective views, which are treated as new information and combined with sample data embedded in the prior distribution via Bayesian updating. The new Bayesian optimal hedging model can help hedgers blend their subjective views with a market-wide or firm-wide consensus in determining optimal hedging positions (ratios).

The rest of paper is organized as follows. Section two outlines the Bayesian portfo-

lio optimization framework and the modification to accommodate an empirical Bayesian approach to subjective views and parameter estimation risk. Section three applies the framework developed in section two to analyze the impact of subjective views on optimal hedging positions (ratios). Section four draws conclusions.

2 Method

The method section is divided into two subsections. The first subsection presents the Bayesian portfolio optimization framework, which provides the theoretical basis for accommodating subjective views and parameter estimation risk within portfolio optimization. The second subsection modifies and customizes the framework with an empirical Bayesian approach, developing a flexible mechanism for accommodating subjective views. Because of the portfolio interpretation of optimal hedging, the method developed in portfolio optimization context can be applied to accommodate subjective views in modelling optimal hedging.

2.1 The Bayesian Portfolio Optimization Framework

Although the mean-variance portfolio optimization model has been the standard theoretical model of normative investment practice, the model has not been widely used by investment practitioners. The major problem with the model is that the mean-variance optimization model is an “estimation error maximizer,” which “over-weights (under-weights) those securities have large (small) estimated returns, negative (positive) correlations, and small (large) variances” in the resulting optimal portfolio (Michaud 1989, p.33–34). Because of the estimation errors in the expected returns, correlations and variances, the mean-variance portfolio optimization, with or without nonnegativity weight constraints, often results in an ill-diversified “optimized” portfolio, which puts extreme positions in a few assets and performs relatively poorly out-of-sample compared to an equal-weighted portfolio or a market-capitalized portfolio.

Concerned with parameter estimation risk within the portfolio optimization framework, researchers (Brown 1979; Jorion 1985; Frost and Savarino 1986; Jorion 1986; Black and Litterman 1992; Pastor 2000) have applied the principles of the Bayesian decision theory to modify the standard mean-variance portfolio optimization model. The new Bayesian model determines the optimal portfolio weight vector via expected utility maximization (mean-variance optimization) conditioned on the predictive distribution (predictive expectation vector and predictive covariance matrix) of asset returns. Mathematically, the optimization problem can be stated as follows,

$$\begin{aligned}
& \max_{w \in W} E_{\theta} \{ E_{r_{t+1}|\theta} [U(w' * r_{t+1})] \} \\
& = \max_{w \in W} \int_{\Theta} \left\{ \int_{R_{t+1}} U(w' r_{t+1}) P(r_{t+1}|\theta) dr_{t+1} \right\} p(\theta|X_t, I_t) d\theta \\
& = \max_{w \in W} \int_{R_{t+1}} U(w' r_{t+1}) \left[\int_{\Theta} P(r_{t+1}|\theta) p(\theta|X_t, I_t) \right] dr_{t+1} \\
& = \max_{w \in W} \int_{R_{t+1}} U(w' r_{t+1}) p(r_{t+1}|X_t, I_t) dr_{t+1}
\end{aligned} \tag{1}$$

where w and W denote, respectively, the portfolio weight vector and its domain, and θ and Θ denote, respectively, the parameters of asset returns and its domain. The parameters of

asset returns include the expected return vector, μ , and the covariance matrix, Σ . Note that r_{t+1} and R_{t+1} denote, respectively, the vector of realized asset returns in period $t + 1$ and its domain, and I_t and X_t denote, respectively, the prior belief and new information. With the pure Bayesian approach adopted, the prior distribution is calibrated with subjective views and the new information is sample data. With an empirical approach adopted, the prior distribution is calibrated with sample data and subjective views represent the new information. $U(\cdot)$, $E(\cdot)$, $p(\cdot)$ denote, respectively, a utility function, expectation function and probability distribution function.

Bayesian decision theory (Gelman et al. 2004, p.85–87) requires that the predictive probability distribution function of asset returns in period $t + 1$, $p(r_{t+1}|X_t, I_t)$, be computed from the integration of the product of the likelihood function $P(r_{t+1}|\theta)$ and posterior distribution $p(\theta|X_t, I_t)$ over uncertain parameters, θ . The posterior distribution is proportional to the product of the prior pdf, $P(\theta|I_t)$, and the likelihood function, $p(X|\theta)$.

With a mean-variance utility function adopted, the optimization above can be simplified into mean-variance optimization conditioned on the predictive expectation vector and predictive covariance matrix of asset returns (Polson and Tew 2000). Mathematically, the optimization problem can be re-stated as,

$$\begin{aligned} & \max_{q \in W} \int_R U(w'r_{t+1})p(r_{t+1}|X_t, I_t)dr_{t+1} \\ & = \max_{w \in W} w'E[r_{t+1}|X_t, I_t] - \frac{\tau}{2}w'Var[r_{t+1}|X_t, I_t]w \\ & = \max_{w \in W} w'\mu_{t+1}|(X_t, I_t) - \frac{\tau}{2}w'\Sigma_{t+1}|(X_t, I_t)w \end{aligned} \quad (2)$$

where τ denotes the absolute Arrow-Pratt risk aversion coefficient, and $\mu_{t+1}|(X_t, I_t)$ and $\Sigma_{t+1}|(X_t, I_t)$ denote, respectively, the predictive expectation vector and predictive covariance matrix of asset returns.

2.2 Empirical Modification

To accommodate subjective views in realistic scenarios, the Bayesian mean-variance portfolio optimization model above should be modified and customized with an empirical Bayesian approach. The empirical Bayesian approach calibrates the prior distribution with sample data, which are assumed to represent the market consensus,¹ and updates the prior with practitioners' subjective views on asset returns. Bayesian updating yields the predictive expectation vector and predictive covariance matrix of asset returns, which can be input into the mean-variance optimization to determine the optimal portfolio weight vector.

2.2.1 Specification and Calibration of the Prior Distribution

Assume asset returns follow a multivariate normal distribution. Mathematically it is

$$r|\mu, \Sigma \sim N_n(\mu, \Sigma) \quad (3)$$

where μ and Σ denote, respectively, the $n \times 1$ vector of the expectations and the $n \times n$ covariance matrix of the asset returns, where n assets are considered.

¹The representation of market consensus with sample data only serves as a simplifying assumption here to facilitate the model derivation in this section, and obviously the sample data is only one of the inputs in the market consensus formation.

Although the empirical Bayesian approach requires calibrating the prior distribution with sample data, it does not mandate any specific form of prior specification. The exact form of the prior specification chosen by practitioners should reflect and be consistent with practitioners’ perception of parameter estimation risk. For example, concerned with estimation risk in both expected returns and the covariance matrix of asset returns, Frost and Savarino (1986) choose a Normal-Wishart prior, which can accommodate parameter estimation errors in both the expectation vector and the covariance matrix of asset returns.² With an empirical Bayesian approach adopted, assuming that all assets have “identical expected returns, variances, and pairwise correlation coefficients,” Frost and Savarino calibrate the prior distribution with the average expected return, average variance, and average correlation coefficient of all assets in the population. Obviously the Normal-Wishart prior can also be calibrated with different expectation vector and covariance matrix estimates based on sample data. If practitioners are mainly concerned with estimation risk in the expectation vector of asset returns, they may adopt the normal prior of the expectation vector of asset returns proposed by Black and Litterman (1992). The normal prior can accommodate estimation errors in the expectation vector of asset returns, but ignores the estimation risk in the covariance matrix of asset returns. Black and Litterman’s prior specification is appropriate for accommodating parameter estimation risk in a portfolio optimization context because, compared with the covariance matrix of asset returns, expected returns are much more difficult to estimate (Merton 1980), and estimation errors in expected returns may have a much bigger impact on the resulting optimal portfolio (Jobson and Korkie 1980; Jorion 1986; Best and Grauer 1991). Black and Litterman calibrate the prior of expected returns to implied equilibrium asset returns, assuming the market is in an equilibrium condition.³ Given its theoretical appeal and analytical simplicity, Black and Litterman’s prior is adopted in this study and mathematically it is

$$\mu \sim N(\mu_0, \lambda^{-1}\Sigma) \quad (4)$$

where the expectation vector of asset returns, μ , is assumed to follow an n-dimensional multivariate normal distribution with expectation vector μ_0 and covariance matrix $\lambda^{-1}\Sigma$. As noted above, the prior distribution should be calibrated with the sample data. The practitioner’s confidence in the prior belief concerning expected asset returns is measured with λ . For example, if a practitioner’s confidence level associated with the prior is set comparable to the market volatility measured by the covariance matrix of asset returns, the practitioner should set the confidence level to be one, i.e., $\lambda = 1$, while if the practitioner feels more confident of the prior, perhaps, the practitioner can set the confidence level to be two or even a larger number. The bigger λ is the higher the confidence level.⁴

2.2.2 Expression of the View(s)

Practitioners’ subjective views on asset returns can be categorized into two classes: “absolute views” and “relative views”. Absolute views refer to practitioners’ opinions concerning the absolute performance of assets. For example, while the market consensus estimated from sample data indicates that the annual rate of return on asset A should be 4%, a

²See Frost and Savarino (1986), Polson and Tew (2000) and Pastor (2000) for details of applications of various Normal-Wishart priors.

³See Bevan and Winkelmann (1998) for the details on the rationale and computation of implied equilibrium returns.

⁴The confidence level associated with the prior should be a function of the number of sample observations and the perceived precision of the econometric or statistical model used in the parameter estimation.

practitioner estimates it to be 5% with a confidence level of 90%.⁵ The relative views refer to practitioners' opinions concerning the relative performance of two or more assets. For example, while the market consensus estimated from sample data indicates that asset C should outperform asset B by 2% annually, a practitioner estimates it to be 1% with a confidence level of 75%. Notice that a subjective view, whether absolute or relative, should be paired with a corresponding confidence level. The subjective views can be conveniently expressed in a matrix format as,

$$P * \mu \sim N(q, \Omega) \quad (5)$$

where P is a $k \times n$ matrix, each row of which represents a view on expected asset returns and selects the assets involved in the view and specifies the relative weights of these assets. Notice that $k \leq n$ because the number of views must be less than number of assets. The vector of expected asset returns is μ , q is a $k \times 1$ vector, each entry of which specifies the expected return of the portfolio specified according to a view, and Ω measures the average confidence level associate with all the views and is decomposed as $\Omega = \kappa^{-1}P\Sigma P'$, where κ measures the average confidence level of the views. For example, if the practitioner's uncertainty concerning views in aggregate is set comparable to market volatility, the confidence coefficient κ would equal one, alternatively, if the practitioner's average confidence level of the views is 90%, the λ should then be set to 90% of the market volatility.

2.2.3 Bayesian Updating and the Predictive Pair

The subjective views expressed by equation (5) can be intuitively viewed as an under-determined linear system (Peressini, Sullivan, and Uhl 1988, p.142), which imposes certain linear restrictions on the expected asset returns. The system is under-determined because the number of views must be less than number of assets, i.e., $k \leq n$. With the linear system interpretation of the subjective views, the subjective views can be re-stated as

$$q = P\mu - v \text{ where } v \sim N_p(0, \Omega). \quad (6)$$

The minimum-norm solution of the system is

$$\hat{\mu} = P'(PP')^{-1}q \text{ with estimation volatility as } (P'\Omega^{-1}P)^{-1}. \quad (7)$$

The linear restrictions imposed by the subjective views, treated as new information, should be combined with the prior belief through Bayesian updating, which renders the posterior distribution of expected asset returns, $\mu|X_t$. With the Bayesian updating formula applied, the posterior distribution of expected asset returns should be a multivariate normal distribution with expectation vector

$$\begin{aligned} & [(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\lambda^{-1}\Sigma)^{-1}\mu_0 + P'\Omega^{-1}P\hat{\mu}] \\ & = [(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\lambda^{-1}\Sigma)^{-1}\mu_0 + P'\Omega^{-1}q] \end{aligned} \quad (8)$$

and covariance matrix ⁶

$$[(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}. \quad (9)$$

The expectation vector of the posterior distribution of the expected asset returns (equation (8)) can be intuitively interpreted as a weighted average of the prior belief and the subjective

⁵Similar to the confidence level associated with the prior, the confidence level associated with a subjective view is also set with respect to the covariance matrix of asset returns.

⁶See Appendix A for the derivation of the predictive expectation vector and the predictive covariance matrix of asset returns.

views, where the prior belief μ_0 is weighted with the inverse of the covariance matrix in the prior, $(\lambda^{-1}\Sigma)^{-1}$, and the subjective views, represented by the minimum-norm solution, is weighted with the inverse of the estimation volatility, $P'\Omega^{-1}P$.

The posterior distribution of expected returns can be further integrated to compute the predictive distribution of asset returns as

$$p(r_{t+1}|X_t) = \int p(r_{t+1}|\mu, \Sigma)\pi(\mu|X_t)d\mu. \quad (10)$$

According to Bayesian theory (Gelman et al. 2004, p.85–87), the predictive distribution of asset returns should follow a multivariate normal distribution with expectation vector

$$\begin{aligned} E[r_{t+1}|X_t] &= [(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\lambda^{-1}\Sigma)^{-1}\mu_0 + P'\Omega^{-1}q] \\ &= \mu_0 + [(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}P'\Omega^{-1}[q - P\mu_0] \\ &= \mu_0 + [\lambda^{-1}\Sigma - \lambda^{-1}\Sigma P'(\Omega + P\lambda^{-1}\Sigma P')^{-1}P\lambda^{-1}\Sigma]P'\Omega^{-1}[q - P\mu_0] \end{aligned} \quad (11)$$

and covariance matrix

$$\begin{aligned} Cov[r_{t+1}|X_t] &= E[\Sigma|X_t] + Var[\mu|X_t] = \Sigma + [(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1} \\ &= \Sigma + \lambda^{-1}\Sigma - \lambda^{-2}\Sigma P'[\Omega + \lambda^{-1}P\Sigma P']^{-1}P\Sigma \end{aligned} \quad (12)$$

Finally, the predictive expectation vector and the predictive covariance matrix of asset returns should be input into the mean-variance optimization in determining the optimal hedging position (ratio) as a special case.

3 Application

With the empirical Bayesian framework developed above, a practitioner can consistently adjust his/her optimal hedging position (ratio) based on his/her view(s). Some simplifying assumptions are made here for ease of exposition and explicitness of results when these assumptions do not impair the features of the model that we seek to emphasize. Specifically, we assume that a hedger, within a one-period static setup, maximizes his/her mean-variance utility function by hedging his/her spot position with futures contracts, and that the hedger has no other investment opportunities and does not borrow or lend, and that the markets are frictionless, which means no commissions, no margin requirements and no lumpiness due to standardization of futures contracts. In addition, the returns of assets under consideration are assumed to follow a multivariate normal distribution.

The application of the framework is illustrated with the following two examples. The first example illustrates the impact on the optimal hedging position (ratio) of a single absolute view on the expected futures return, highlighting the different optimal hedging positions (ratios) suggested by the alternative optimal hedging models. The second example analyzes the impact on the optimal hedging position (ratio) of a relative view expressed on the expected basis.

3.1 A Bet on the Expected Futures Returns

Although the unbiasedness hypothesis of futures prices is still a controversial issue, most researchers agree that even if premia do exist, they are likely small for most commodity futures (Fama and French 1987). Nonetheless, practitioners often have opinions concerning

the directional move of prices in these futures markets. Survey studies noted in the introduction to this paper have shown practitioners sometimes actively take positions based on their market views; so how should practitioners adjust their optimal hedging positions (ratios) according to their directional views on the markets? This example illustrates how a hedger should adjust his/her optimal hedging position (ratio) according to a single directional view on the futures market by applying the empirical Bayesian model developed in the previous section.

For comparison, we first present the standard mean-variance optimal hedging model in a typical one-period static setup. The optimal hedging model has been derived with mean-variance maximization of end-period wealth, profit/loss (price change in units of the underlying asset), and portfolio rate of return. The different specifications are associated with different objectives of the hedgers and different assumptions concerning the stochastic process of asset prices (Myers and Thompson 1989), but the difference is not important for the purpose of model derivation because under different assumptions, we can still derive identical Bayesian models, whose parameters can be estimated according to practitioner's assumptions of the hedging objective and/or the stochastic process of asset prices or returns. Here we adopt the optimal hedging model based on mean-variance maximization of profit/loss, which has been widely used in the optimal hedging literature.

The first-order differences of the commodity spot and futures prices are assumed to follow a multivariate normal distribution with expectation vector

$$\mu = \begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} \quad (13)$$

and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} \quad (14)$$

where with some notation manipulation, μ and Σ now denote, respectively, the expectation vector and the covariance matrix of the price differences, and subscripts s and f denote, respectively, the commodity spot and futures.

The hedger is assumed to have a fixed long spot position at the beginning of the period and expects to offset the spot position at the end of the period. The hedger wants to reduce the price risk exposure by shorting a futures position. The mean-variance maximization of the end-period profit/loss is

$$\begin{aligned} & \max_{X_f} E[r_s Y_s + r_f Y_f] - \frac{\tau}{2} \text{Var}[r_s Y_s + r_f Y_f] \\ & = \max_{Y_f} [Y_s \quad Y_f] \begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} - \frac{\tau}{2} [Y_s \quad Y_f] \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} \begin{bmatrix} Y_s \\ Y_f \end{bmatrix} \end{aligned} \quad (15)$$

The optimal hedging position can be obtained from the first order condition of the mean-variance maximization as,

$$Y_f^* = \frac{\mu_f}{\tau \sigma_f^2} - Y_s \frac{\sigma_{sf}}{\sigma_f^2} \quad (16)$$

which consists of a speculative component, $\frac{\mu_f}{\tau \sigma_f^2}$, and a pure hedging component, $-Y_s \frac{\sigma_{sf}}{\sigma_f^2}$.

In practice, the optimal hedging position (ratio) is often computed with the standard PCE procedure, which directly substitutes sample estimates for the true but unknown parameters. The empirical optimal hedging position according to the PCE procedure is

$$Y_f^* = \frac{\hat{\mu}_f}{\tau \hat{\sigma}_f^2} - Y_s \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2} \quad (17)$$

Alternatively, the empirical optimal hedging ratio according to the PCE procedure is

$$\frac{Y_f^*}{Y_s} = \frac{\hat{\mu}_f}{\tau \hat{\sigma}_f^2 Y_s} - \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2} \quad (18)$$

If the market consensus is that the futures price is unbiased, i.e., $\hat{\mu}_f = 0$, under the PCE procedure, the empirical optimal hedging position is

$$Y_f^* = -Y_s \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2}. \quad (19)$$

Now suppose that the hedger instead has a directional view on the expected price difference of futures, forecasting that the expected price difference of the futures is q instead of 0. According to the PCE procedure, the hedger should ignore the subjective view and hedge according to equation (19). Alternatively, the hedger may naively replace the $\hat{\mu}_f$ estimate with q in equation (17) and obtain an alternative optimal hedging position

$$Y_f^* = \frac{q}{\tau \hat{\sigma}_f^2} - Y_s \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2} \quad (20)$$

However this procedure completely ignores sample information concerning the expected price difference of futures, and puts extreme confidence on the hedger's view. The method often leads to an extreme hedging position. Lence and Hayes (1994b, 1994a) characterize this method as the Perfect Parameter Information (PPI) method due to putting extreme confidence on the hedger's view.

Intuitively, the hedger would rather blend both sample information and the view together according to their relative confidence levels in determining the optimal hedging position (ratio). The combination of sample data and the view can be achieved through the empirical Bayesian model developed in the method section. With the empirical Bayesian framework applied, the prior distribution of the expectation vector of spot and futures price differences is

$$\begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} \sim N_2 \left(\begin{bmatrix} \mu_{s0} \\ \mu_{f0} \end{bmatrix}, \lambda^{-1} \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} \right) \quad (21)$$

where μ_{s0} and μ_{f0} represent, respectively, the prior beliefs concerning the expected price differences of spot and futures. The prior distribution should be calibrated with sample data. For analytical simplicity, we simply substitute sample estimates for the parameters of the prior distribution. The subjective view is paired with its corresponding confidence level

$$\mu_f \sim N\left(q, \frac{1}{\kappa} \sigma_f^2\right) \quad (22)$$

which indicates that the hedger forecasts the expected futures difference as q with a confidence level $\frac{1}{\kappa} \sigma_f^2$, where κ measure the confidence level of the view with respect to the volatility of the futures price difference, σ_f^2 . Putting it into the matrix form of equation (5), we obtain $P = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $q = [q]$ and $\Omega = \frac{1}{\kappa} [\sigma_f^2]$.

Inputting these parameters into the derived predictive expectation vector and the predictive covariance matrix of asset returns (equation (11) and equation (12)) in the previous section, with some algebraic manipulation, we can obtain the predictive expectation vector of the price differences of spot and futures

$$\mu_{t+1} | X_t, I_t = \begin{bmatrix} \mu_{s0} + \frac{\kappa}{\kappa+\lambda} (q - \mu_{f0}) \frac{\sigma_{sf}}{\sigma_f^2} \\ \mu_{f0} + \frac{\kappa}{\kappa+\lambda} (q - \mu_{f0}) \end{bmatrix} \quad (23)$$

and the predictive covariance matrix of the price differences of spot and futures ⁷

$$\Sigma_{t+1}|X_t, I_t = \begin{bmatrix} (1 + \frac{1}{\lambda})\sigma_s^2 - \frac{\kappa}{\lambda(\kappa+\lambda)}\rho^2\sigma_s^2 & (1 + \frac{1}{\lambda})\sigma_{sf} - \frac{\kappa}{\lambda(\kappa+\lambda)}\sigma_{sf} \\ (1 + \frac{1}{\lambda})\sigma_{sf} - \frac{\kappa}{\lambda(\kappa+\lambda)}\sigma_{sf} & (1 + \frac{1}{\lambda})\sigma_f^2 - \frac{\kappa}{\lambda(\kappa+\lambda)}\sigma_f^2 \end{bmatrix} \quad (24)$$

Substituting the elements of the predictive expectation vector and the predictive covariance matrix for the first-order condition of mean-variance maximization (equation (16)), we can determine the optimal hedging position, which yields

$$Y_f^* = \frac{\mu_{f0} + \frac{\kappa}{\kappa+\lambda}(q - \mu_{f0})}{\frac{\kappa+\lambda+1}{\kappa+\lambda}\tau\sigma_f^2} - Y_s \frac{\sigma_{sf}}{\sigma_f^2}. \quad (25)$$

Because the market consensus assumes that the expected difference of futures is zero (unbiased), i.e., $\mu_{f0} = 0$, the new optimal hedging position is

$$Y_f^* = \frac{\kappa}{\kappa + \lambda + 1} \cdot \frac{q}{\tau\hat{\sigma}_f^2} - Y_s \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2}. \quad (26)$$

The Bayesian optimal hedging position consists of a pure hedging component and a speculative component. The pure hedging component is the same as that in the standard optimal hedging model with PCE or PPI implementation because the Bayesian updating changes proportionally the covariance and the variance of the spot and futures price differences (see equation (24)), which renders an unchanged pure hedging component (minimum-variance hedging ratio).

The speculative component is a weighted average of the PCE and PPI implementation of the standard optimal hedging model. The weight coefficient $\frac{\kappa}{\kappa+\lambda+1}$ ranges from zero to one according to the relative confidence of the hedger on the view with respect to the sample estimate. The empirical Bayesian model, similar to Lence and Hayes model (1994a, 1994b), nests two extreme implementations of the optimal hedging model. As the hedger feels more and more confident about the market consensus, i.e., $\lambda \rightarrow \infty$, the weight coefficient approaches zero, consequently, the hedger should ignore the view and the optimal hedging position shrinks to the PCE result. As the hedger feels more and more confident about the view, i.e., $\kappa \rightarrow \infty$, the weight coefficient approaches one, consequently, the hedger should hedge according his/her view, and the speculative component of the optimal hedging position converts to the PPI result $\frac{q}{\tau\hat{\sigma}_f^2}$. The relation between the weight coefficient and confidence level of the view is illustrated in figure (1), where we assume that the confidence level associated with the prior is one, i.e., $\lambda = 1$.⁸

In more realistic hedging scenarios, hedgers would blend the sample estimates with their view according to the relative confidence levels. In most cases, the hedger should feel more confident of the consensus estimated from the sample data than the view. Suppose that based on his/her experience of the past performance of sample estimates of the spot and futures price difference, the hedger sets the confidence level of the prior as $\lambda = 1$, which means that the confidence level associate with the prior is set comparable to the market volatility measured by the covariance matrix of asset returns. Now, suppose that the hedger sets the confidence level of the views as $\kappa = 0.2$ because he/she feels much less

⁷See Appendix B for the derivation of the predictive expectation vector and the predictive covariance matrix of asset returns.

⁸In reality, the hedger never can be certain of the expected futures returns, thus the result is hypothetical and unrealistic, but it does provide insight into how the hedger should weigh the view according to its relative confidence level.

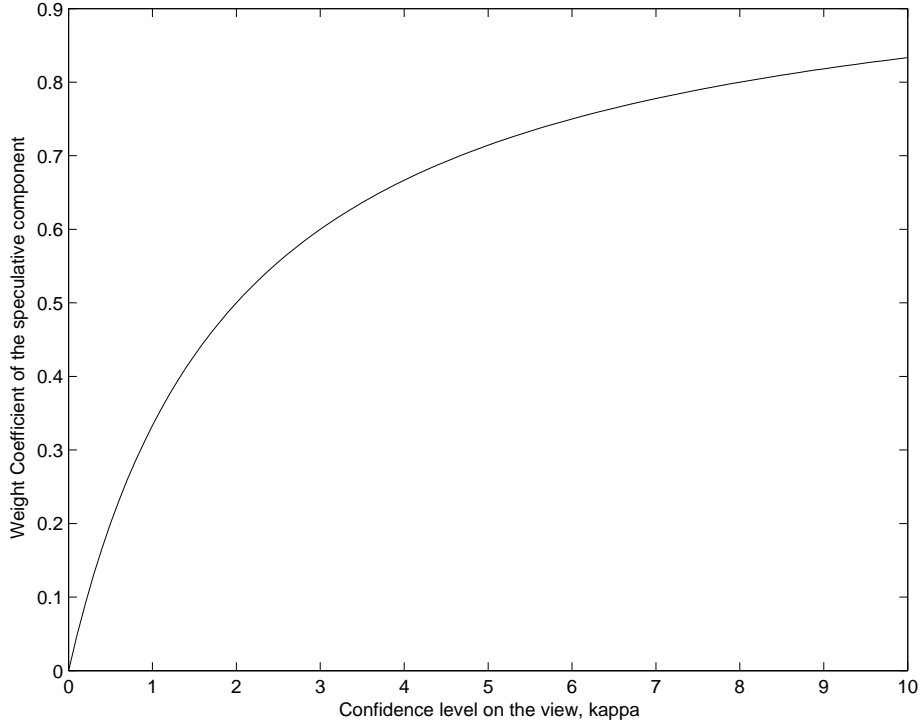


Figure 1: The weight coefficient of the speculative component of the optimal hedging position, assuming the confidence level associated with sample data is one, i.e., $\lambda = 1$ and an absolute bet on expected futures price difference.

confident of the view. With these additional assumptions, the Bayesian optimal hedging position is

$$Y_f^* = \frac{1}{11} \times \frac{q}{\tau \hat{\sigma}_f^2} - Y_s \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2} \quad (27)$$

Thus concerned with the speculative component, the hedger should compromise between the two extreme positions suggested by the PCE and PPI implementation, but tilt toward the PCE result because the hedger feels more confident of the prior estimated from the sample data. Notice that the speculative component also depends on the magnitude of the view, which is measured by q .

The impact of a single subjective view concerning expected futures price difference can also be illustrated with the following realistic hedging scenario: on 02/19/2004, a corn hedger in the North Central region of Illinois has a spot position and expects to offset the position on 07/08/2004, 20 weeks later. To reduce price risk exposure, the hedger wants to hedge his/her spot position with CBOT July 2004 corn futures. The sample data used to estimate parameters are corn spot prices for the North Central region of Illinois for the post-harvest periods from 1975 to 2003, and prices of the corresponding CBOT July corn futures.⁹ The sample estimate of the expectation vector and the covariance matrix of price differences (in dollar per bushel) for 20 week holding periods are

$$\begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} = \begin{bmatrix} 0.1489 \\ 0.0066 \end{bmatrix} \text{ and } \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} = \begin{bmatrix} 0.1020 & 0.0977 \\ 0.0977 & 0.1064 \end{bmatrix} \quad (28)$$

The sample estimate of the expectation vector is used to calibrate the prior distribution. Now suppose that the hedger speculates that the expected price difference of futures is

⁹The dataset is obtained from AgMAS project at University of Illinois at Urbana-Champaign.

\$0.25/bushel instead of the sample estimate, \$0.0066/bushel. The confidence levels of the prior and the views are set to be 1 and 0.2 as before.¹⁰ Then the optimal hedging position is $0.0624\frac{1}{\tau} - 0.9185Y_s$ according to the PCE procedure (equation (17)), and $2.3496\frac{1}{\tau} - 0.9185Y_s$ according to the PPI procedure (equation(20)), but it is $0.2418\frac{1}{\tau} - 0.9185Y_s$ according to the new Bayesian model (equation(25)). With different procedures, the speculative component of the optimal hedging position varies substantially though the pure hedging component remains unchanged.

The new Bayesian optimal hedging model suggests that while the hedger's single view on the expected futures price difference has no impact on the pure hedging component of the optimal hedging position, it can significantly alter the speculative component of the optimal hedging position. The magnitude of this change depends on how far the magnitude of the subjective view deviates away from the consensus and the relative confidence level of the subjective view with respect to the consensus.

3.2 A Bet on the Expected Basis

The second example illustrates the impact on the optimal hedging position of a single relative view expressed on the expected basis. With the same model setup used in the first example, we now assume that although historical records project the basis to be b_1 , the hedger expects it to be \tilde{b}_1 . This hedging scenario is a standard basis arbitrage with the addition of a confidence level on the view. The view is a relative view concerning the performance of spot and futures due to the following transformation

$$\begin{aligned} E_0[b_1] - b_0 &= E_0[S_1 - F_1] - [S_0 - F_0] \\ &= E_0[S_1 - S_0] - E_0[F_1 - F_0] \\ &= \mu_s - \mu_f \end{aligned} \tag{29}$$

Equation (29) shows that speculation on the expected basis, $E_0[b_1]$, can be transformed into a bet on the relative performance of spot and futures. As in the first example, the view can be expressed in a matrix format (equation (5)) as

$$[1 \ -1] \begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} \sim N(\tilde{b}_1, \Omega) \tag{30}$$

where $P = [1 \ -1]$, and $q = \tilde{b}_1$, and $\Omega = \kappa^{-1}P\Sigma P' = \kappa^{-1}(\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2)$.

Inputting the prior distribution and the linear restriction imposed by the view, we can compute the predictive expectation vector and the predictive covariance matrix of price differences of spot and futures based on equation (11) and equation (12) in the previous section. The predictive expectation vector is

$$\mu_{t+1}|X_t, I_t = \begin{bmatrix} \mu_{s0} + \frac{\lambda\kappa}{\kappa\lambda + \lambda^2} \frac{\sigma_s^2 - \sigma_{sf}}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} (\tilde{b}_1 - b_1) \\ \mu_{f0} + \frac{\lambda\kappa}{\kappa\lambda + \lambda^2} \frac{\sigma_{sf} - \sigma_f^2}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} (\tilde{b}_1 - b_1) \end{bmatrix} \tag{31}$$

¹⁰The prior calibration and subjective view formation can be achieved by using more sophisticated econometric and other types of methods, but for illustration purpose, the simplest sample estimates are used in this example.

and the predictive covariance matrix is ¹¹

$$\Sigma_{t+1}|X_t, I_t = \begin{bmatrix} (1 + \frac{1}{\lambda})\sigma_s^2 - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_s^2 - \sigma_{sf})^2}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} & (1 + \frac{1}{\lambda})\sigma_{sf} - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2)}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} \\ (1 + \frac{1}{\lambda})\sigma_{sf} - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2)}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} & (1 + \frac{1}{\lambda})\sigma_f^2 - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_{sf} - \sigma_f^2)^2}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} \end{bmatrix} \quad (32)$$

The optimal hedging position is determined via mean-variance optimization (equation (17)) conditioned on the predictive expectation vector and predictive covariance matrix,

$$Y_f^* = \frac{\mu_f b_0 + \frac{\lambda\kappa}{(\kappa\lambda + \lambda^2)} \frac{\sigma_{sf} - \sigma_f^2}{\sigma_s^2 - \sigma_{sf} + \sigma_f^2} (\tilde{b}_1 - b_1)}{\tau[(1 + \frac{1}{\lambda})\sigma_f^2 - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_{sf} - \sigma_f^2)^2}{\sigma_s^2 - \sigma_{sf} + \sigma_f^2}]} - Y_s \frac{(1 + \frac{1}{\lambda})\sigma_{sf} - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2)}{\sigma_s^2 - \sigma_{sf} + \sigma_f^2}}{(1 + \frac{1}{\lambda})\sigma_f^2 - \frac{\kappa}{\lambda(\kappa+\lambda)} \frac{(\sigma_{sf} - \sigma_f^2)^2}{\sigma_s^2 - \sigma_{sf} + \sigma_f^2}} \quad (33)$$

The optimal hedging position (equation (33)) indicates that unlike the absolute view in the first example, the relative view affects both the speculative and pure hedging components of the optimal hedging position, while the magnitude of the view $\tilde{b}_1 - b_1$ only affects the pure hedging component.

The impact of the view can only be analyzed with empirical data due to the optimal hedging position's lack of analytical tractability. We assume the same hedging scenario and sample data used in the first example. In addition, we assume that on 02/19/2004, the hedger observes the July basis at \$ -0.1975/bushel. Although historical data (average basis) project that the July basis on 07/08/2004 would be -0.13676\$/bushel, the hedger expects it to be -0.1\$/bushel.

The impact of the view on the speculative and pure hedging components should be analyzed separately. The relative view on the expected basis has only a modest impact on the pure hedging component. The relation between the pure hedging component and the confidence level of the view is illustrated in Figure (2). The relation shows that because of the view, the pure hedging component (minimum-variance optimal hedging ratio) calculated from the new Bayesian model is always higher than that in the standard model implemented with PCE procedure, which is 0.9185. The pure hedging component increases gradually as the confidence level of the view increases, but is bounded by an upper limit at about 0.96.

The view does have a considerable impact on the speculative component of the optimal hedging position. The relation between the speculative component and the confidence level of the view is illustrated in Figure (3). While the speculative component in the standard optimal hedging model implemented with PCE procedure is $0.06243\frac{1}{\tau}$, the relation shows that as the confidence level of the view passes a certain threshold, the speculative component of the optimal hedging position first turns from a long position prescribed by PCE procedure to a short position, and that as the confidence level increases, the size of the short speculative component also increases but eventually levels out.

The Bayesian model suggests that the relative view on the expected basis can change both the pure hedging and the speculative components of the optimal hedging position, but the impact of the view is more significant on the speculative component than on the pure hedging component. The magnitude of the changes also depends on the magnitude of the view and the relative confidence level of the view with respect to the prior belief.

¹¹See Appendix C for the derivation of the predictive expectation vector and the predictive covariance matrix of asset returns.

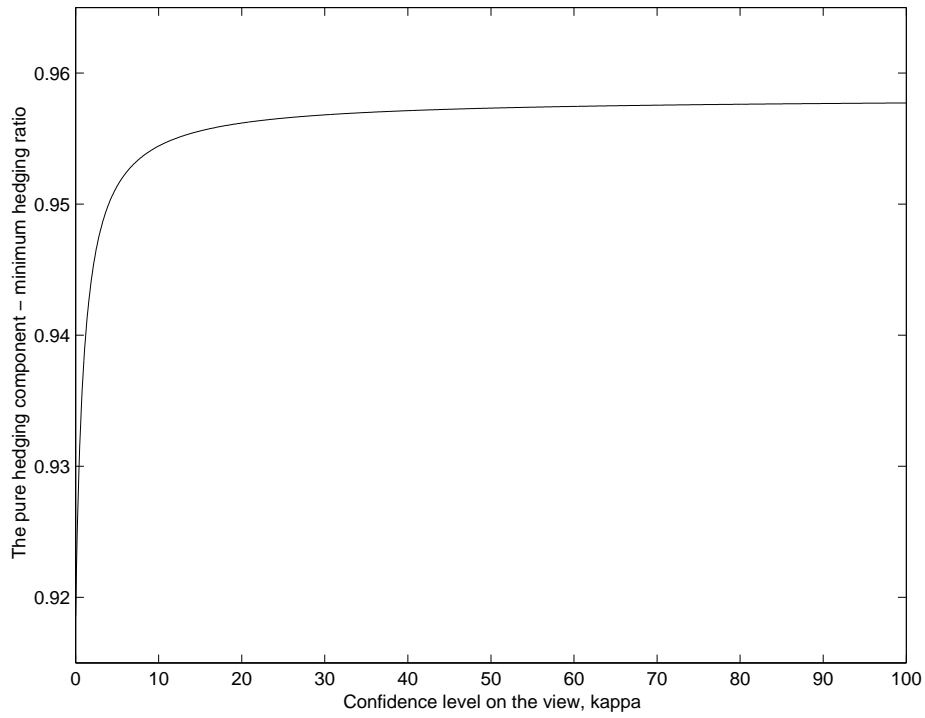


Figure 2: The pure hedging component of the optimal hedging position, assuming the confidence level associated with sample data is one, i.e., $\lambda = 1$ and a bet on the expected basis.

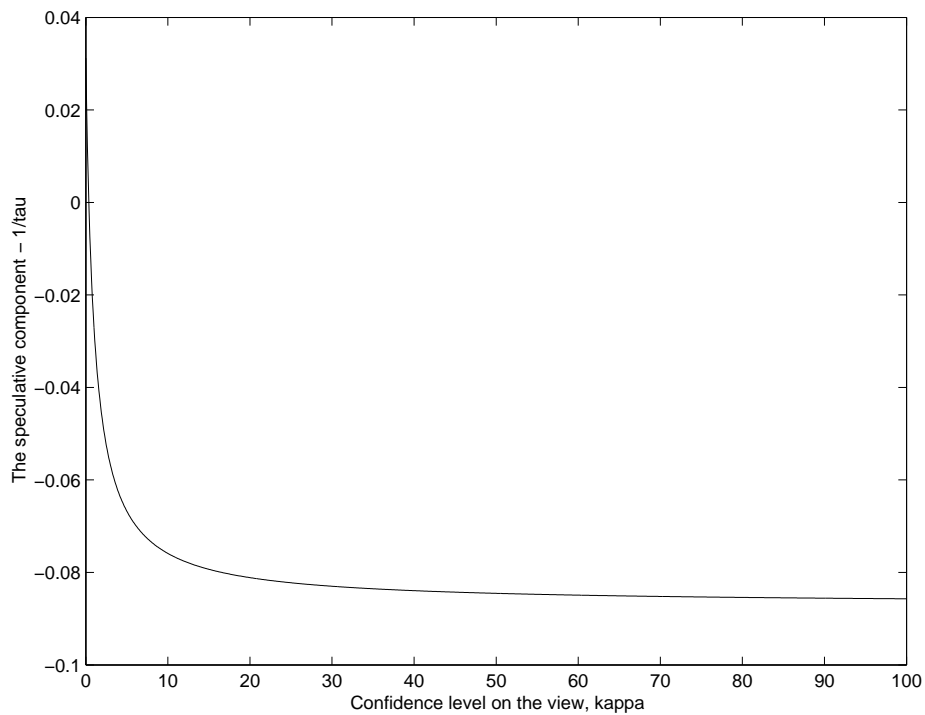


Figure 3: The speculative component of the optimal hedging position, assuming the confidence level associated with sample data is one, i.e., $\lambda = 1$ and a bet on the expected basis.

4 Conclusions

Given the widespread use of subjective views in hedging practice and the potential economic benefit of selective hedging, the lack of accommodation of subjective views in the optimal hedging model is a serious problem and could hamper the model's application in risk management practice. Because of their inherent link, the subjective view problem should be tackled together with the parameter estimation risk problem within the optimal hedging framework. The Bayesian optimal hedging model developed by Lence and Hayes (1994b, 1994a) can effectively tackle the parameter estimation problem, but can only accommodate subjective views under some restrictive and unrealistic assumptions, which requires calibrating the whole prior distribution with subjective views, while hedgers may have views on only one or two parameters.

Instead, with an empirical Bayesian approach adopted, this study proposes an alternative Bayesian optimal hedging model, in which a hedger can adjust his/her optimal hedging position (ratio) according to his/her view(s) on the expected returns of assets under consideration. Like Lence and Hayes' model (1994b, 1994a), the optimal hedging position is also determined by the mean-variance optimization conditioned on the predictive expectation vector and predictive covariance matrix of asset returns, but unlike their model, the number and type of subjective views that can be expressed are quite flexible because of the adoption of an empirical Bayesian approach. The impact of subjective view(s) on the optimal hedging position is illustrated with two examples. The first example analyzes how a hedger should adjust his/her optimal hedging position (ratio) according to a single absolute view concerning the expected futures price difference. The results show that although the view has no impact on the pure hedging component of the optimal hedging position, it can significantly alter the speculative component of the optimal hedging position, and the magnitude of this change depends on how far the magnitude of the subjective view deviates away from the consensus and the relative confidence level of the subjective view with respect to the consensus. The second example analyzes the impact of a single relative view expressed on the expected basis. The results show that although the view changes both the pure hedging and the speculative components of the optimal hedging position, the impact of the view is more significant on the speculative component than on the pure hedging component. The magnitude of the changes also depends on the magnitude of the view and the relative confidence level of the view with respect to the prior belief.

The empirical Bayesian optimal hedging model provides practitioners with a theoretically intuitive yet quantitatively rigorous framework to blend the subjective view(s) and the market consensus estimated from sample data according to their relative confidence levels. The model also can shed some insight on the well-documented phenomenon of the wide cross-sectional and time-series variation in actual hedging positions, which is difficult to explain on the basis of different parameter estimates, different risk-aversion coefficients, and/or different optimal hedging models. The substantial variation in hedging positions has to be at least partially attributed to the selective hedging activities based on their subjective views.

The Bayesian optimal hedging model developed in this study can be further generalized to accommodate subjective views expressed on the covariance matrix of asset returns because practitioners can also have subjective views on the covariance matrix of asset returns given the widespread use of conditional volatility models. Moreover, some of the restrictive and simplifying assumptions used in the model derivation can be relaxed to accommodate more realistic hedging scenarios. For example, the model can be extended into a multi-period hedging context, quantity risk and transaction costs may also be incorporated into the model.

References

- Adam, Tim R., and Chitru S. Fernando. 2003. "Are There Speculative Components in Corporate Hedging and Do They Add Value?" Working Paper, Business School, Hong Kong University of Science and Technology, Hong Kong.
- Bertoli, Roberto, Carl Zulauf, Scott H. Irwin, Thomas E. Jackson, and Darrel L. Good. 1999. "The Marketing Style of Advisory Services for Corn and Soybeans in 1995." AgMAS Project Research Report 1999–02, Department of Agricultural and Consumer Economics, University of Illinois at Urbana-Champaign, IL.
- Best, Michael J., and Robert R. Grauer. 1991. "On the Sensitivity of Mean-Variance-Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results." *Review of Financial Studies* 4 (2): 315–342.
- Bevan, Andrew, and Kurt Winkelmann. 1998. "Using the Black-Litterman Global Asset Allocation Model: Three Years of Practical Experience." Technical Report, Goldman Sachs, Global Fixed Income Portfolio Strategy, Fixed Income Research.
- Black, Fischer, and Robert Litterman. 1990. "Asset Allocation: Combining Investors Views with Market Equilibrium." Technical Report, Fixed Income Research, Goldman & Sachs Company.
- . 1992. "Global Portfolio Optimization." *Financial Analyst Journal* 48 (5): 28–43.
- Bodnar, Gordon M., Gregory S. Hayt, and Richard C. Marston. 1996. "1995 Wharton Survey of Derivatives Usage by US Non-Financial Firms." *Financial Management* 25 (4): 113–133.
- . 1998. "1998 Wharton Survey of Financial Risk Management by US Non-financial Firms." *Financial Management* 27 (4): 70–91.
- Brown, Gregory W., Peter R. Crabb, and David Haushalter. 2002. "Are Firms Successful at Selective Hedging?" Working Paper, Kenan-Flagler Business School, University of North Carolina at Chapel Hill, NC.
- Brown, Stephen. J. 1979. Chapter Optimal Portfolio Choice under Uncertainty: A Bayesian Approach of *Estimation Risk and Optimal Portfolio Choice*, edited by Vijay S. Bawa, Stephen J. Brown, and Roger W. Klein. North-Holland, Amsterdam.
- Dolde, Walter. 1993. "The Trajectory of Corporate Financial Risk Management." *Journal of Applied Corporate Finance* 6 (3): 33–41.
- Fama, Eugene F., and Kenneth R. French. 1987. "Commodity Future Prices: Some Evidence on Forecast Power, Premiums, and the Theory of Storage." *Journal of Business* 60 (1): 55–73.
- Frost, Peter A., and James E. Savarino. 1986. "An Empirical Bayes Approach to Efficient Portfolio." *Journal of Financial and Quantitative Analysis* 21 (3): 293–305.
- Gelman, Andrew, John B. Carlin, Hal S. Stern, and Donald B. Rubin, eds. 2004. *Bayesian Data Analysis*. 2nd. ed. Texts in Statistical Science. Chapman & Hall/CRC.
- Greene, William H. 2002. *Econometric Analysis*. 5th edition. Prentice Hall.
- He, Guangliang, and Robert Litterman. 1999. "The Intuition Behind Black-Litterman Model Portfolios." Technical Report, Investment Management Research. Goldman and Sachs Company.

- Irwin, Scott H., Joao Martines-Filho, and Darrel L. Good. 2003. "The Pricing Performance of Market Advisory Services in Corn and Soybeans Over 1995-2001." AgMAS Project Research Report 2003-05, Department of Agricultural and Consumer Economics, University of Illinois at Urbana-Champaign, IL.
- Jobson, J.D, and B. Korkie. 1980. "Estimation for Markowitz Efficient Portfolios." *Journal of the American Statistical Association* 75 (371): 544-554.
- Jorion, Philippe. 1985. "International Portfolio Diversification with Estimation Risk." *Journal of Business* 58 (3): 259-278.
- . 1986. "Bayes-Stein Estimator for Portfolio Analysis." *Journal of Financial and Quantitative Analysis* 21 (3): 279-292.
- Lence, Sergio H., and Dermot J. Hayes. 1994a. "The Empirical Minimum-Variance Hedge." *American Journal of Agricultural Economics* 76 (1): 94-104.
- . 1994b. "Parameter-Based Decision Making Under Estimation Risk: An Application to Futures Trading." *Journal of Finance* 49 (1): 345-357.
- Martines-Filho, Joao, Brian G. Stark, Silvina M. Cabrini, Scott H. Irwin, Darrel L. Good, Wei Shi, Ricky L. Webber, Lewis A. Hagedorn, and Steven L. Williams. 2003a. "Advisory Service Marketing Profiles for Corn Over 1995-2000." AgMAS Project Research Report 2003-03, Department of Agricultural and Consumer Economics, University of Illinois at Urbana-Champaign, IL.
- . 2003b. "Advisory Service Marketing Profiles for Soybeans Over 1995-2000." AgMAS Project Research Report 2003-03, Department of Agricultural and Consumer Economics, University of Illinois at Urbana-Champaign, IL.
- McNew, Kevin, and Wesley N. Musser. 2002. "Farmer Forward Pricing Behavior: Evidence from Marketing Clubs." *Agricultural and Resource Economics Review* 31 (2): 200-210.
- Merton, Robert C. 1980. "On Estimating the Expected Return on the Market." *Journal of Financial Economics* 8 (4): 323-361.
- Michaud, Richard O. 1989. "The Markowitz Optimization Enigma; Is 'Optimized' Optimal?" *Financial Analyst Journal* 45 (1): 31-42.
- Myers, Robert J., and Stanley R. Thompson. 1989. "Generalized Optimal Hedge Ratio Estimation." *American Journal of Agricultural Economics* 71 (4): 858-868.
- Pastor, Lubos. 2000. "Portfolio Selection and Asset Pricing Models." *Journal of Finance* 55 (1): 179-223.
- Peressini, Anthony L., Francis E. Sullivan, and J. Jerry Uhl. 1988. *The Mathematics of Nonlinear Programming*. Undergraduate Texts in Mathematics. Springer-Verlag.
- Polson, Nicholas G., and Bernard V. Tew. 2000. "Bayesian Portfolio Selection: An Empirical Analysis of the S&P 500 Index 1970 - 1996." *Journal of Business and Economic Statistics* 18 (2): 164-173.
- Stulz, Rene M. 1996. "Rethinking Risk Management." *Journal of Applied Corporate Finance* 9 (3): 8-24.
- Working, Holbrook. 1962. "New Concepts Concerning Futures Markets and Prices." *American Economic Review* 52 (3): 432-459.

Appendix A

In this appendix, we derive step by step the Bayesian portfolio optimization model in the method section. First, the prior distribution of μ , the expected asset returns, is combined with subjective views concerning these expected returns via Bayesian updating, which yields a posterior distribution of the expected asset returns. Second, the posterior distribution is integrated into a predictive distribution of asset returns. Third, the predictive distribution is input into the expected utility function to determine the weight vector of the optimal portfolio.

The prior distribution of the expectation vector of asset returns, μ , is assumed to be a multivariate normal distribution (equation (4)), whose probability distribution function is

$$p(\mu|I_t) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} \exp\left[-\frac{1}{2}(\mu - \mu_0)'(\lambda^{-1}\Sigma)^{-1}(\mu - \mu_0)\right] \quad (34)$$

The restrictions that the subjective views impose on the μ (equation (5)) can be expressed into following condition probability distribution function

$$p(P\mu - q|\mu) = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}} \exp\left[-\frac{1}{2}(P\mu - q)'\Omega^{-1}(P\mu - q)\right] \quad (35)$$

Then the prior distribution of μ is combined with the subjective views via a Bayesian updating formula (Gelman et al. 2004, p.8), which yields the posterior distribution of the expected asset returns as

$$\begin{aligned} p(\mu|I_t, X_t) &\propto P(\mu|I_t) \cdot P(P\mu - q|\mu) \\ &\propto \exp\left[-\frac{1}{2}(\mu - \mu_0)'(\lambda^{-1}\Sigma)^{-1}(\mu - \mu_0) - \frac{1}{2}(P\mu - q)'\Omega^{-1}(P\mu - q)\right] \end{aligned} \quad (36)$$

where \propto denotes proportionality. $P(\mu|I_t)$ and $P(P\mu - q|\mu)$ denote, respectively, the prior distribution of μ and restrictions imposed by the subjective views on μ . Equation (36) shows that the posterior probability distribution function of μ is an exponential of a quadratic form in μ . Therefore if we define $H = (\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P$ and $C = (\lambda^{-1}\Sigma)^{-1}\mu_0 + P'\Omega^{-1}q$ and $A = \mu_0'(\lambda^{-1}\Sigma)^{-1}\mu_0 + q'\Omega^{-1}q$,¹² we can complete the quadratic form and pull out constant factors to obtain

$$\begin{aligned} p(\mu|I_t, X_t) &\propto \exp\left\{-\frac{1}{2}\left[\mu'H\mu - (C\mu)' - (C\mu) + A\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[\mu'H'H^{-1}H\mu - (CH^{-1}H\mu)' - (CH^{-1}H\mu) + A\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[(H\mu - C)'H^{-1}(H\mu - C) - C'H^{-1}C + A\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[-C'H^{-1}C + A\right]\right\} \exp\left\{-\frac{1}{2}\left[(H\mu - C)'H^{-1}(H\mu - C)\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[(H\mu - C)'H^{-1}(H\mu - C)\right]\right\} \\ &\propto \exp\left\{-\frac{1}{2}\left[(\mu - H^{-1}C)'H(\mu - H^{-1}C)\right]\right\} \end{aligned} \quad (37)$$

¹²Notice that H is a symmetric matrix so that $H = H'$.

The equation (37) shows that the posterior probability distribution function of μ is also a multivariate normal distribution with expectation vector,

$$\begin{aligned} H^{-1}C &= \left[(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P \right]^{-1} (\lambda^{-1}\Sigma)^{-1}\mu_0 + P'\Omega^{-1}q \\ &= \mu_0 + \left[\lambda^{-1}\Sigma - \lambda^{-1}\Sigma P'(\Omega + P\lambda^{-1}\Sigma P')^{-1}P\lambda^{-1}\Sigma \right] P'\Omega^{-1}[q - P\mu_0] \end{aligned} \quad (38)$$

and covariance matrix as

$$H = \left[(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P \right] \quad (39)$$

The posterior distribution of expected returns can be further integrated to compute the predictive distribution of asset returns as shown in equation (10). According to Bayesian theory (Gelman et al. 2004, p.85–87), the predictive distribution of asset returns should be a multivariate normal distribution with expectation vector,

$$\mu|X_t, I_t = \left[(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P \right]^{-1} (\lambda^{-1}\Sigma)^{-1}\mu_0 + P'\Omega^{-1}q \quad (40)$$

and covariance matrix,

$$\begin{aligned} \Sigma|X_t, I_t &= \Sigma + [(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1} \\ &= \Sigma + \lambda^{-1}\Sigma - \lambda^{-2}\Sigma P' [\Omega + \lambda^{-1}P\Sigma P^{-1}] P\Sigma \end{aligned} \quad (41)$$

where $[(\lambda^{-1}\Sigma)^{-1} + P'\Omega^{-1}P]^{-1} = \lambda^{-1}\Sigma - \lambda^{-1}\Sigma P' [\Omega + P\lambda^{-1}\Sigma P']^{-1} P\lambda^{-1}\Sigma$ is due to a matrix updating formula (Greene 2002, p.32).

The predictive expectation vector and the predictive covariance matrix of asset returns then can be input into the mean-variance expected utility function to determine the weight vector of the optimal portfolio as shown in equation (2).

Appendix B

In this appendix, we apply the Bayesian portfolio optimization framework to the futures hedging scenario detailed in the subsection ‘‘A Bet on the Expected Futures Returns’’ of the application section. With the prior distribution of μ (equation (21)) and the subjective view (equation (22)) plugged into formulas for the predictive expectation vector (equation 11) and the predictive covariance matrix (equation 12) of asset returns, we compute the predictive expectation vector of the cash and futures returns

$$\begin{aligned} \mu_{t+1}|X_t, I_t &= \mu_0 + \left[\lambda^{-1}\Sigma - \lambda^{-1}\Sigma P' [\kappa^{-1}\sigma_f^2 + \lambda^{-1}\sigma_f^2]^{-1} P\lambda^{-1}\Sigma \right] P'\Omega^{-1}(q - P\mu_0) \\ &= \mu_0 + \frac{1}{\lambda}\Sigma P'\Omega^{-1}(q - P\mu_0) - \frac{\kappa}{\lambda(\kappa + \lambda)} \cdot \frac{1}{\sigma_f^2}\Sigma P' P\Sigma P'\Omega^{-1}(q - P\mu_0) \\ &= \mu_0 + \frac{\kappa}{\lambda} \cdot \frac{1}{\sigma_f^2}\Sigma P'(q - P\mu_0) - \frac{\kappa^2}{\lambda(\kappa + \lambda)} \cdot \frac{1}{\sigma_f^2}\Sigma P'(q - P\mu_0) \\ &= \mu_0 + \frac{\kappa}{\kappa + \lambda} \cdot \frac{1}{\sigma_f^2}\Sigma P'(q - P\mu_0) \\ &= \begin{bmatrix} \mu_{s0} \\ \mu_{f0} \end{bmatrix} + \frac{\kappa}{\kappa + \lambda} \cdot \frac{1}{\sigma_f^2}(q - P\mu_0) \begin{bmatrix} \sigma_{sf} \\ \sigma_f^2 \end{bmatrix} \\ &= \begin{bmatrix} \mu_{s0} + \frac{\kappa}{\kappa + \lambda}(q - P\mu_0)\frac{\sigma_{sf}}{\sigma_f^2} \\ \mu_{f0} + \frac{\kappa}{\kappa + \lambda}(q - P\mu_0) \end{bmatrix} \end{aligned} \quad (42)$$

and the predictive covariance matrix of the cash and futures returns

$$\begin{aligned}
\Sigma|X_t, I_t &= \Sigma + \lambda^{-1}\Sigma + \frac{\kappa}{\lambda(\kappa + \lambda)} \frac{1}{\sigma_f^2} \Sigma P' P \Sigma \\
&= \left(1 + \frac{1}{\lambda}\right) \Sigma - \frac{\kappa}{\lambda(\kappa + \lambda)} \frac{1}{\sigma_f^2} \begin{bmatrix} \sigma_{sf}^2 & \sigma_{sf}\sigma_f^2 \\ \sigma_{sf}\sigma_f^2 & \sigma_f^4 \end{bmatrix} \\
&= \begin{bmatrix} \left(1 + \frac{1}{\lambda}\right)\sigma_s^2 - \frac{\kappa}{\lambda(\kappa + \lambda)}\rho^2\sigma_s^2 & \left(1 + \frac{1}{\lambda}\right)\sigma_{sf} - \frac{\kappa}{\lambda(\kappa + \lambda)}\sigma_{sf} \\ \left(1 + \frac{1}{\lambda}\right)\sigma_{sf} - \frac{\kappa}{\lambda(\kappa + \lambda)}\sigma_{sf} & \left(1 + \frac{1}{\lambda}\right)\sigma_f^2 - \frac{\kappa}{\lambda(\kappa + \lambda)}\sigma_f^2 \end{bmatrix}
\end{aligned} \tag{43}$$

The predictive expectation vector and the predictive covariance matrix of cash and futures returns are then input into the standard optimal hedging model (equation (16)) to determine the new optimal hedging ratio (equation 25 and equation (26)).

Appendix C

In this appendix, we apply the empirical Bayesian portfolio optimization model to the futures hedging scenario detailed in the subsection ‘‘A Bet on the Expected Basis’’ in the application section. With the prior distribution of μ (equation (21)) and the subjective view (equation (30)) plugged into the formulas for the predictive expectation vector (equation 11) and the predictive covariance matrix (equation 12) of asset returns, we compute the predictive expectation vector of the cash and futures returns

$$\begin{aligned}
\mu_{t+1}|X_t, I_t &= \mu_0 + \left[\lambda^{-1}\Sigma - \lambda^{-1}\Sigma P' [(\kappa^{-1} + \lambda^{-1})(\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2)]^{-1} P \lambda^{-1}\Sigma \right] P' \Omega^{-1}(\tilde{b}_1 - b_1) \\
&= \mu_0 + \left[\frac{1}{\lambda}\Sigma - \frac{\kappa}{\lambda(\kappa + \lambda)} \cdot \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} \Sigma P' P \Sigma \right] P' \kappa \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} (\tilde{b}_1 - b_1) \\
&= \mu_0 + \frac{\kappa}{\lambda} \cdot \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} (\tilde{b}_1 - b_1) \Sigma P' - \frac{\kappa^2}{\lambda(\kappa + \lambda)} (\tilde{b}_1 - b_1) \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} \Sigma P' \\
&= \mu_0 + \frac{\kappa\lambda}{\lambda(\kappa + \lambda)} \cdot \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} (\tilde{b}_1 - b_1) \Sigma P' \\
&= \begin{bmatrix} \mu_{s0} \\ \mu_{f0} \end{bmatrix} + \frac{\kappa\lambda}{\lambda(\kappa + \lambda)} \cdot \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} (\tilde{b}_1 - b_1) \begin{bmatrix} \sigma_s^2 - \sigma_{sf} \\ \sigma_{sf} - \sigma_f^2 \end{bmatrix} \\
&= \begin{bmatrix} \mu_{s0} + \frac{\kappa\lambda}{\lambda(\kappa + \lambda)} \cdot \frac{\sigma_s^2 - \sigma_{sf}}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} (\tilde{b}_1 - b_1) \\ \mu_{f0} + \frac{\kappa\lambda}{\lambda(\kappa + \lambda)} \cdot \frac{\sigma_{sf} - \sigma_f^2}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} (\tilde{b}_1 - b_1) \end{bmatrix}
\end{aligned} \tag{44}$$

and the predictive covariance matrix of the cash and futures returns

$$\begin{aligned}
\Sigma|X_t, I_t &= \Sigma + \lambda^{-1}\Sigma + \frac{1}{\lambda^2} \Sigma P' [(\kappa^{-1} + \lambda^{-1})(\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2)]^{-1} P \Sigma \\
&= \left(1 + \frac{1}{\lambda}\right) \Sigma - \frac{\kappa}{\kappa\lambda + \lambda^2} \cdot \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} \Sigma P' P \Sigma \\
&= \left(1 + \frac{1}{\lambda}\right) \Sigma - \frac{\kappa}{\kappa\lambda + \lambda^2} \cdot \frac{1}{\sigma_f^2 - 2\sigma_{sf} + \sigma_s^2} \begin{bmatrix} (\sigma_s^2 - \sigma_{sf})^2 & (\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2) \\ (\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2) & (\sigma_{sf} - \sigma_f^2)^2 \end{bmatrix} \\
&= \begin{bmatrix} \left(1 + \frac{1}{\lambda}\right)\sigma_s^2 - \frac{\kappa}{\lambda(\kappa + \lambda)} \cdot \frac{(\sigma_s^2 - \sigma_{sf})^2}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} & \left(1 + \frac{1}{\lambda}\right)\sigma_{sf} - \frac{\kappa}{\lambda(\kappa + \lambda)} \cdot \frac{(\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2)}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} \\ \left(1 + \frac{1}{\lambda}\right)\sigma_{sf} - \frac{\kappa}{\lambda(\kappa + \lambda)} \cdot \frac{(\sigma_s^2 - \sigma_{sf})(\sigma_{sf} - \sigma_f^2)}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} & \left(1 + \frac{1}{\lambda}\right)\sigma_f^2 - \frac{\kappa}{\lambda(\kappa + \lambda)} \cdot \frac{(\sigma_{sf} - \sigma_f^2)^2}{\sigma_s^2 - 2\sigma_{sf} + \sigma_f^2} \end{bmatrix}
\end{aligned} \tag{45}$$

Then as before, the predictive expectation vector and the predictive covariance matrix of cash and futures returns are input into the standard optimal hedging model (equation (16)) to determine the new optimal hedging ratio (equation 33).