Increasing the Accuracy of Option Pricing by Using Implied Parameters Related to Higher Moments

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The inaccuracy of the Black-Scholes formula arises from two aspects: the formula is for European options while most real option contracts are American; the formula is based on the assumption that underlying asset prices follow a lognormal distribution while in the real world asset prices cannot be described well by a lognormal distribution. We develop an American option pricing model that allows non-normality. The theoretical basis of the model is Gaussian quadrature and dynamic programming. The usual binomial and trinomial models are special cases. We use the Jarrow-Rudd formula and the relaxed binomial and trinomial tree models to imply the parameters related to the higher moments. The results demonstrate that using implied parameters related to the higher moments is more accurate than the restricted binomial and trinomial models that are commonly used.

Keywords: option pricing, volatility smile, Edgeworth series, Gaussian Quadrature, relaxed binomial and trinomial tree models.

Introduction

Accurate option valuation formulas are highly demanded by participants in financial and commodity markets. Option valuation formulas are used to predict prices and related risk and then to make trading and risk management decisions. Among the formulas currently used in practice, the Black-Scholes option valuation formula has been the most popular since its publication in 1973. While ease of use is its advantage, the inaccuracy of the formula is widely recognized. At least for short maturity options, the biases in the Black-Scholes are serious (Backus et al.). Due to this inaccuracy, traders either regularly adjust the volatility parameter in the formula by personal experience or use alternatives such as binomial tree models and jump-diffusion models. However, Dumas, Fleming, and Whaley argue that there is no evidence showing superior performance of those models over the Black-Scholes formula.

Black-Scholes is an analytic price formula for European options, the rights that cannot be exercised before expiration. In the real world, however, most option contracts are American which can be exercised at any time during the life of the options. Besides this American-European difference, the biases in the Black-Scholes formula also arise from the lognormal assumption underlying the model. Lognormal distribution is a commonly used assumption in financial modeling about asset price movements. Under such an assumption, the distribution of the underlying asset price is characterized by a single volatility parameter. Thus the different options on a given asset, though corresponding to different strike prices, must imply the same volatility. However, this is not true in the real world. If equations between real option prices and the Black-Scholes formula are established and the volatility is solved from each of the equations, different solutions will be obtained for equations corresponding to different strike prices. This phenomenon is known as “volatility smile”. In other words, if we substitute a constant volatility into the formula to predict option prices with different strike prices but the same maturity, the biases will be serious. Therefore, lognormal is a convenient assumption for modeling asset price distributions but far from an accurate approximation for the real situation. Many studies have shown that the prices of assets, either commodities or financial securities, generally do not follow a lognormal distribution.

To make the Black-Scholes formula applicable to American options, some analytic American option valuation formulas have been developed. However, in general, analytic formulas are not suitable for American options, except for a few special types, namely, American calls on an asset that pays discrete dividends, and perpetual American call and put options. In practice, another widely used options pricing model, especially for American options, is the binomial tree model. The original version of the model was developed by Cox, Ross, and Rubinstein in 1979. The binomial tree model is a numerical method. The most important advantage of numerical methods over analytic valuations lies in the ability to provide a numerical solution to problems where analytic solutions do not exist or are difficult to evaluate. While the binomial tree model is applicable to American options, it is not necessarily more accurate than the Black-Scholes formula. In fact, the asymptotic limit of the binomial formula is the Black-Scholes formula. This result is valid for all versions of the binomial model (Kwok). This means that the biases due to the lognormal distribution assumption can occur in the binomial tree model. When such biases are significant enough, the binomial tree model may be less accurate than the Black-Scholes formula even though being used to American options.

Another way to correct the biases in the Black-Scholes is to correct the assumption about the underlying distribution by using the Edgeworth series. It is well known in mathematical and physical work that functions can often be usefully expressed as a series of terms such as powers of the variable (Taylor's series) or trigonometrical functions (Fourier's series). In statistics, although neither of these forms is very suitable for probability density functions, there are some similar counterparts. One such approximate series is the Edgeworth series. The basic idea of the Edgeworth series is that the probability density function of an unknown distribution can be expressed as a known probability density function, say lognormal density function, plus a series of linear combination of higher moments. The higher the moments included in the series, the more accurate the approximate series. Generally, when the fourth moment is included, the accuracy of the approximation can be satisfactory enough (Johnson et al.).

The Edgeworth series provides a potential way to model asset price distributions as well as the behavior of option prices based on the asset more accurately. The representative work to explore this potential is the approximate option valuation model developed by Jarrow and Rudd in 1983. Analogous to the style that the Edgeworth series can be expressed as an extension of lognormal density function, the Jarrow-Rudd approximate formula can be written as an extension of the Black-Scholes formula. The parameter related to the highest moment in the Jarrow-Rudd is the fourth cumulant. While this direction of modeling is theoretically attractive, it is rare to be used in practice. The difficulty of the work on this line is the determination of the cumulant parameters. The only to be determined in a lognormal style option pricing model, such as the Black-Scholes, is the volatility while there are at least three parameters to be determined in the Jarrow-Rudd. The volatility parameter in the Black-Scholes can either be estimated by using historical data or be implied by fitting the formula to real option prices. For the Jarrow-Rudd approximate formula, there is no reliable method available to calculate the three parameters simultaneously yet. To imply the three parameters by fitting a single formula to real option prices, additional constraints about the relationship among the parameters are necessary. The information about such constraints is not available. The investigation by Barton and Dennis in 1952 provides some boundary conditions on the values of the third and fourth cumulants for the density functions to be non-negative, but the exact relationship among the parameters is still not clear. Moreover, the Jarrow-Rudd approximation is still a European option pricing formula. So it is not suitable for American options.

The purpose of this study is to provide a method that is more accurate than the Black-Scholes formula yet easy to use in practice. The basic idea of this study is that the information about the true distribution is reflected in the real option prices, so that the parameters related to the true distribution can be implied by the real option prices though it is impossible to model the true distribution of the underlying asset price exactly. Multiple implied parameters should be helpful to predict the option prices more accurately than a single parameter based on the lognormal distribution assumption. The method of implying the volatility parameter by the Black-Scholes formula is already widely used in both practice and theoretical research. The novelty of this study is to imply multiple parameters in both analytic and numerical tree options pricing models. The binomial and trinomial tree models are commonly used in practice. However, the single parameter in the models is the result of some arbitrary constraints on the original parameters of the models and is usually just implied by the Black-Scholes formula. The performance of more parameters implied directly by the tree models has not, if ever, been investigated very much.

Models and Method

The option pricing models used in this study are the Black-Scholes formula, the Jarrow-Rudd approximation formula, the binomial and trinomial tree models. The Black-Scholes formula is a single parameter formula and is commonly used to imply the volatility parameter. Implied volatility is the volatility that is implicit in the prices of options. When an explicit pricing formula such as the Black-Scholes formula is available, the quoted prices of the options along with known variables such as interest rates, time to maturity, strike prices and so on can be used in an implicit formula for volatility. The result is called the implied volatility. The basic idea of the Jarrow-Rudd approximate option valuation is that additional parameters can adjust the error that is not captured by the single parameter. The tree models are originally multi-parameter models but reduced to the single parameter models by adding artificial constraints on the original parameters in the models. For the convenience of discussion, we call the binomial or trinomial trees with such artificial constraints "restricted" binomial or trinomial trees. Though the trees, especially the binomial trees, are also widely used in practice as the Black-Scholes, they are rarely, if ever, used to imply the volatility parameter in the models. A natural explanation for this interesting phenomenon is that the artificial constraints on the original parameters may conflict with the true relationship among the parameters so that the implied volatility restricted by the artificial constraints would be biased seriously. An intuition about the binomial and trinomial trees is that a tree model without artificial constraints may be more accurate than the model with such constraints. Again, for convenience, we call the tree models without artificial constraints "relaxed" binomial or trinomial trees. With the constraints relaxed, a tree model becomes a multi-parameter model and thus may possess the properties proposed by the Jarrow-Rudd model. On the other hand, compared with the Jarrow-Rudd model, while keeping the multi-parameter characteristic, the relaxed trees have an important advantage over the Jarrow-Rudd formula. After some artificial constraints are relaxed, there are still two constraints in the tree models that can be guaranteed to be correct while there is no information about the relationship among the parameters in the Jarrow-Rudd formula available. Moreover, the relaxed trees are applicable to both European and American options while the Jarrow-Rudd model is only applicable to European options. We next briefly introduce the concepts and models involved in this study, and then illustrate the method adopted in this study.

Lognormal Distribution and Edgeworth Series

A commonly used assumption in financial modeling is that price process of asset is governed by a Geometric Brownian motion:

$$S(t) = e^{X(t)} \qquad t \geq 0 \tag{1}$$

where $S(t)$ is asset price at time t, $X(t)$ denotes a Brownian motion with drift parameter $\mu \geq 0$ and variance parameter σ^2. This is equivalent to say that, at given time t, $S(t)$ is lognormally distributed with a probability density function

$$g(s) = \frac{1}{s\,\sigma\sqrt{2\pi t}} \exp\left(-\frac{(\ln s - \mu t)^2}{2\sigma^2 t} \right) \qquad s > 0 \tag{2}$$

Although, in general, asset price does not follow a lognormal distribution and the true distribution is unknown, the true probability density function of the asset price can be approximated as an Edgeworth series containing a lognormal density function $g(s)$ (Jarrow and Rudd 1983):

$$f(s) = g(s) + \frac{(k_2 - k_2(G))}{2!}\frac{d^2 g(s)}{ds^2} - \frac{(k_3 - k_3(G))}{3!}\frac{d^3 g(s)}{ds^3}$$
$$+ \frac{((k_4 - k_4(G)) + 3(k_2 - k_2(G))^2)}{4!}\frac{d^4 g(s)}{ds^4} + \varepsilon(s) \tag{3}$$

where $f(s)$ is the true probability density function of the asset price, $\varepsilon(s)$ is the residual error, and the cumulants k_js are defined by the following equations:

$$\alpha_j = \int_{-\infty}^{\infty} s^j f(s)\,ds$$

$$\mu_j = \int_{-\infty}^{\infty} (s - \alpha_1)^j f(s)\,ds$$

$$j = 1, 2, 3, 4$$

$$k_2 = \mu_2 \quad k_3 = \mu_3 \quad k_4 = \mu_4 - 3\mu_2 \qquad (4)$$

The cumulants $k_j(G)$s are defined similarly to k_js by changing the density function from $f(s)$ to $g(s)$.

Thus if an Edgeworth series containing cumulants up to the fourth is used to approximate the true density function, the error is $\epsilon(s)$. If the lognormal density function is used as an approximation for the true density function, the error will be a series of higher cumulants plus $\epsilon(s)$. As long as the true distribution is not lognormal, the Edgeworth series is a more accurate approximation for the true probability density function than the lognormal density function.

The Black-Scholes and Jarrow-Rudd Formulas

Based on the assumption that the probability density function of underlying asset prices is represented by equation (2), the call and put options on futures contracts can be valued by a futures version of the Black-Scholes formula:

$$c = e^{-rT}[F_0 N(d_1) - XN(d_2)]$$

$$p = e^{-rT}[XN(-d_2) - F_0 N(-d_1)]$$

where

$$d_1 = \frac{\ln(F_0 / X) + \sigma^2 T / 2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0 / X) - \sigma^2 T / 2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \qquad (5)$$

and c is the call option price, p is the put option price. $N(x)$ is the cumulative probability distribution function for a standard normal variable. F_0 is the current futures price, X is the strike price, r is the continuously compounded risk-free rate, σ is the futures price volatility, and T is the time to maturity of the option.

On the other hand, based on the assumption that the underlying unknown probability density function can be approximated by (3), call options can be evaluated by an approximated formula derived by Jarrow and Rudd:

$$c = c(G) + e^{-rT}\frac{(k_2 - k_2(G))}{2!}g(X) - e^{-rT}\frac{(k_3 - k_3(G))}{3!}\frac{dg(X)}{dS} + e^{-rT}\frac{((k_4 - k_4(G)) + 3(k_2 - k_2(G))^2)}{4!}\frac{d^2 g(X)}{dS^2} + \epsilon(X) \qquad (6)$$

where $c(G)$, with G representing the underlying density function $g(s)$, is the Black-Scholes formula for the call option, and other symbols are defined as before. The Jarrow-Rudd formula for the put option price can be obtained by the Put-Call parity, that is

$$p = c + e^{-rT}(X - F_0) \tag{7}$$

Restricted Binomial and Trinomial Trees

To illustrate the pricing mechanism of the binomial tree model, consider a portfolio consisting of a long position in $\mathbf{D}$ contracts of the futures valued at F_0 and a short position in one call option with a value c. The current value of the portfolio is $F_0\mathbf{D} - c$. Suppose the futures price has two possible movements, F_0u ($u > 1$) and F_0d ($d < 1$) respectively, at a future time T. Then the correspondent option values a period later are

$$c_u = max(F_0u - X, 0)$$

and

$$c_d = max(F_0d - X, 0).$$

where c_u is the call option price corresponding to the upward movement of the futures price and c_u to the downward movement. The number $\mathbf{D}$ can be chosen so that the portfolio is riskless no matter which of the two uncertain possibilities realizes. For the portfolio to be riskless, the portfolio should have the same value in either of the possible situations. The portfolio has the same value when

$$F_0u\mathbf{D} - c_u = F_0d\mathbf{D} - c_d$$

or

$$\mathbf{D} = (c_u - c_d)/(F_0u - F_0d).$$

In the absence of arbitrage opportunities, a riskless portfolio must earn the risk-free interest rate r. It follows that

$$F_0\mathbf{D} - c = (F_0u\mathbf{D} - c_u)e^{-rT}$$

or

$$c = F_0\mathbf{D} - (F_0u\mathbf{D} - c_u)e^{-rT}.$$

Substituting Δ into the equation, we have

$$c = [qc_u + (1 - q)c_d]e^{-rT} \tag{8}$$

where

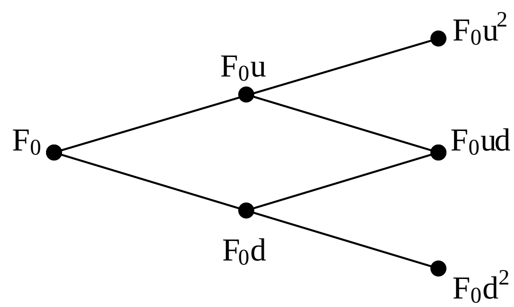
$$q = \frac{e^{rT} - d}{u - d}$$

q can be interpreted as the probability that the futures price moves upward, so the probability that the futures price moves downward is *1 – p.* By a similar way, the binomial tree valuation for a put option is

$$p = [qp_u + (1 - q)p_d]e^{-rT} \tag{9}$$

where p_u and p_d are the put options prices in situations of upward and downward futures prices respectively.

The above one-step binomial tree model can be generalized to a n-step binomial tree model in which a period from current time to future time T is divided into n intervals, each of them has a length $\Delta t = T/n$. As an example, the dynamics of futures prices in a two-step binomial tree is shown in the figure below.



To evaluate European options by the n-step binomial tree model, the parameters u and d (the determination of these parameters will be discussed later) are used to calculate the futures prices step by step until the final step. Then the option prices at the final step are calculated by $max(F_0u^jd^{n-j} – X, 0)$ or $max(X - F_0u^jd^{n-j}, 0)$, depending on they are call options or put options, where j = 0, 1, ···, n. The option prices at the n-1th step are calculated by the prices at the nth step as in the one-step binomial tree. Step by step, by this backward process, the current option price is finally determined. For American options, since they are allowed to be exercised before expiration, at each of the steps, the option value from immediate exercise should be compared with the corresponding European option value to determine whether an early exercise is optimal. For example, at the kth step ($0 < k <$ n), if $F_0u^jd^{k-j} – X$ is greater than the European call option value at that point, it is optimal to exercise the call option immediately. Otherwise, no exercise is optimal.

As mentioned above, q is interpreted as the probability that the futures price goes up. So this probability can be used to calculate the expected futures price at a future time T. Since expected futures price is always equal to current futures price, it follows that

$$qF_0u + (1 – q)F_0d = F_0 \tag{10}$$

or

$$q = \frac{1-d}{u-d}$$

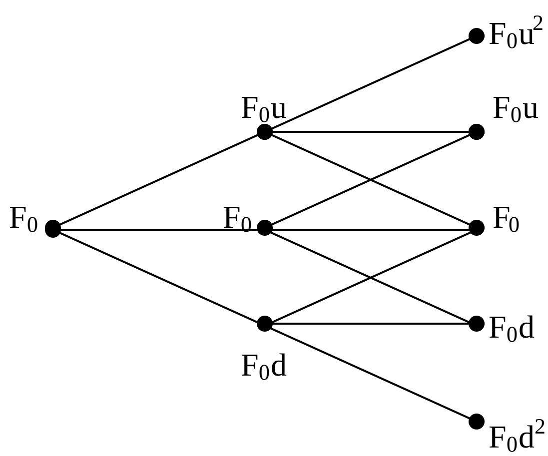
In addition, suppose the variance of the return on the futures price in a short period is the product of a squared volatility parameter σ^2 and the length of the period Δt, we have

$$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2 \Delta t. \tag{11}$$

One solution for the three parameters in these two equations, assuming σ^2 is given, is proposed by Cox, Ross, and Rubinstein(1979). They put an additional constraint $ud = 1$ so that

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

A further extension of the binomial tree is the trinomial tree model introduced by Boyle(1986). Like the binomial tree, the trinomial tree can be used to price both European and American options on a single underlying futures. Because the futures price can move in three directions from a given node, compared with only two in a binomial tree, the number of time steps can be reduced to attain the same accuracy as in the binomial tree. This makes trinomial trees more efficient than binomial trees. There are several ways to choose jump size and move probabilities in a trinomial tree that give the same result when the number of time steps is large. One possibility is to build a trinomial tree where the futures price at each node can go up, stay at the same level, or go down. In that case, the dynamics of futures price can be shown in a two-step trinomial tree as below:



To match the first two moments of the distribution (the mean and variance), the up-and-down jump sizes are:

$$u = e^{\sigma\sqrt{2\Delta t}} \qquad d = e^{-\sigma\sqrt{2\Delta t}}$$

and the probability of going up and down respectively are:

$$q_u = \left(\frac{1 - e^{-s\sqrt{\Delta t/2}}}{e^{s\sqrt{\Delta t/2}} - e^{-s\sqrt{\Delta t/2}}} \right)^2 \text{ and } q_d = \left(\frac{e^{s\sqrt{\Delta t/2}} - 1}{e^{s\sqrt{\Delta t/2}} - e^{-s\sqrt{\Delta t/2}}} \right)^2$$

The probabilities must sum to unity. Thus the probability of staying at the same futures price level is

$$q_m = 1 - q_u - q_d$$

After building the futures price tree, the value of the option can be found in the same way as in the binomial tree by using backward induction.

Constraints on the Parameters from a View of Gaussian Quadrature

An approximate procedure based on Gaussian Quadrature (Miller and Rice) can be used to the constraints on the parameters in a binomial or trinomial tree model. Suppose the probability density function of the futures price is *g(f)*. Then the *k*th moment of the futures price is an integral that can be approximated by a set of futures prices and probabilities such that

$$E(F^k) = \int_{-\infty}^{\infty} F^k g(f) df = \sum_{i=1}^{N} q_i F_i^k \qquad \textit{for } k = 0, 1, 2, \cdots$$

A discrete approximation with N probability-futures pairs can match the first 2N-1 moments exactly by finding q_i and F_i that satisfy the following equations:

$$\sum_{i=1}^{N} q_i = 1$$
$$\sum_{i=1}^{N} q_i F_i = E(F)$$
$$\sum_{i=1}^{N} q_i F_i^2 = E(F^2)$$
$$\vdots$$
$$\sum_{i=1}^{N} q_i F_i^{2N-1} = E(F^{2N-1}).$$

These equations provide the correct constraints on the parameters in a binomial or trinomial tree model, no matter what the true underlying probability distribution is. For a binomial tree model, three equations are needed to fully determine the constraints. For a trinomial tree model, five equations are needed.

Relaxed Binomial and Trinomial Trees

The first two constraints in the above equations are already included in the restricted binomial and trinomial trees mentioned in the previous section. It can be shown (Tian) that if the futures price is lognormally distributed, or equivalently, if F follows a stochastic process

$$dF(t) = F(t)s dz,$$

the third moment of F is

$$E(F^3 \mid F_0) = F_0^3 e^{3s^2\Delta t}.$$

Since the restricted tree model is based on the assumption that the underlying futures price follows a lognormal distribution, a logical constraint matching the third moment should be

$$\sum_{i=1}^{N} q_i F_i^3 = F_0^3 e^{3s^2\Delta t}.$$

In case of a restricted binomial tree, the condition will be:

$$qu^3 + (1-q)d^3 = e^{3s^2\Delta t} = u^6$$

Substituting $u = 1/d$ into the equation and using $q = (1 - d)/(u - d)$, we will have

$$u - u^5 + u^6 - u^9 = 1.$$

This condition obviously cannot always be guaranteed. This implies that the arbitrary condition $u = 1/d$ is in fact conflict with the lognormal assumption underlying the model. According to the above analysis, we see that if we use a restricted tree model to imply the parameters, we in fact assume that the futures price follows a lognormal distribution. However, as mentioned before, this assumption is the source of the biases in option pricing.

How do we determine the constraints on the parameters? Since the true distribution of the futures price is unknown, the values for the moments higher than the first are also unknown. What we can determine by the equations based on the Gaussian Quadrature are only the first two constraints. A good idea may be only keeping the first two constraints that are guaranteed to be correct and leaving the remaining undetermined parameters to be determined by the real market. Real option prices contain the information about the distribution of the underlying futures prices. If a pricing model is correct, the parameters implied by this model should reflect the characteristic of the true distribution. In such a way, more than one parameter will need to be implied by the tree models. Specifically, the parameters q and u can be implied by the binomial tree, leaving the parameter d to be determined by the constraint guaranteed to be true. In the trinomial tree model, three parameters — q_u, u and d — can be implied with the remaining parameters to be determined by the true constraints.

The Method

Implying the parameters in the option pricing models is the basic method of this study. The biases of the Black-Scholes are often stated in terms of a volatility smile. That means if we use a constant volatility to evaluate options with different strike prices in a given day, the sum of the error from each of the options will be large. Thus for a pricing model to be more accurate than the Black-Scholes, there should be a set of parameters such that the forecasting error from using these implied parameters in the model is smaller than the one from the Black-Scholes. Given a model, finding such implied parameters is a non-linear programming problem:

$$\min_{q} \sum_{i=1}^{N} (Call_i - c(F, r, T, X_i, \boldsymbol{q}))^2 + \sum_{j=1}^{M} (Put_j - p(F, r, T, X_j, \boldsymbol{q}))^2$$

where $Call_i$ is the premium for the i-th call option contract, Put_j is the premium for the j-th put option contract, $c(\cdot)$ is the predicted premium by the option pricing model for the call option, $p(\cdot)$ is the predicted premium by the model for the put option, F is the futures price, r is the risk-free interest rate, T is the time of expiration of the option, X_i is the strike price for the i-th call option, X_j is the strike price for the j-th put option, $\boldsymbol{q}$ is the parameter vector to be determined. There are some constraints on $\boldsymbol{q}$ for the relaxed trees, but no constraints available for the Black-Scholes, the Jarrow-Rudd and the restricted tree models.

Once the implied parameters are obtained, the forecasting performance can be measured by out-of-sample fit error. This is a procedure of using yesterday's implied parameters to predict today's option prices. The forecasting error is the measure of the performance of the implied parameters. This study adopts the following measurement for the absolute forecasting error:

$$error_t = \sum_{i=1}^{N} | Call_{it} - c(F_t, r_t, T_t, X_{it}, \boldsymbol{q}_{t-1}) | + \sum_{j=1}^{M} | Put_{jt} - p(F_t, r_t, T_t, X_{jt}, \boldsymbol{q}_{t-1}) | \tag{12}$$

To compare the performances of the models across different commodities, we measure the forecasting errors in terms of per dollar value of the total option premium:

$$\frac{error_t}{\sum_{i=1}^{N} Call_{it} + \sum_{j=1}^{M} Put_{jt}}$$

Data and Procedure

The data that will be used consist of the daily futures prices, the premiums of options on the futures, as well as the corresponding strike prices for each of the three commodities — corn, soybean and wheat. Daily 3-month U.S. treasury bill yields will also be used. Chicago Board of Trade (CBOT) is the source for all the data. The time periods covered is from December 1, 1997, 200 days before the expiration of the 1998 July contracts for all the three commodities, to June 12, 1998, one week before the expiration. There are 4384 observations for corn, 4156 for soybean and 4413 for wheat. For a given futures option, there may be more than 40 contracts available given a business day. For some of these quotes, there are only very thin trades or even no any trade at all. Such quotes are called stale prices because they do not reflect the real situation of demand and supply in the market. Ideally, the data should contain the trade volume for each of the option contracts so we can use the trade volume to filter the stale prices. However, such a data set is not available yet at this stage of time. This insufficiency of data is likely to affect the results quantitatively but not qualitatively.

The empirical procedure consists of two steps. First, by using equation (11), the parameters are implied by minimizing the sum of squared fit errors for six option pricing models for every day. Among the models, the Black-Scholes, restricted binomial and trinomial trees are models with a single volatility parameter. The relaxed binomial tree model has two parameters. In the Jarrow-Rudd and the relaxed trinomial tree models, there are three parameters that need to be implied. For all the restricted and relaxed binomial and trinomial tree models, the number of periods is 5.

Second, as described in equation (12), the implied parameters obtained in the first step are used to calculate the option prices of the next day. The absolute difference between the real option price and the predicted premium by the implied parameters is used to represent the forecasting error for a contract. The summation of all forecasting errors on given futures in a given day is the total forecasting error. To measure the forecasting performance, the magnitude of the premium is taken into account. The forecasting performance is measured by a ratio of the total forecasting error over the sum of the premiums. That is, the forecasting performance is measured by the error in terms of per dollar of premium.

A set of Excel/VBA(Visual Basic for Applications) programs are developed for the empirical procedures.

Results

By using the forecasting error of the Black-Scholes formula as the basis, the relative forecasting errors of other models with the three commodities are shown in the three figures. For corn (see Figure1), by using yesterday's implied parameters, all the models other than the Black-Scholes can be used to forecast the option prices more accurate than the Black-Scholes formula. Among the models, the most accurate one is the relaxed binomial and trinomial trees, whose forecasting errors are less than 20% of the Black-Scholes'. The restricted trees, though more accurate than the Black-Scholes and the Jarrow-Rudd, are not as accurate as the relaxed counterparts. Between the two restricted trees, the restricted binomial is more accurate than the restricted trinomial tree. The two relaxed trees perform about as well as each other.

Figure 2 shows that the relative accuracy of the models is reversed for soybeans. The two analytic option pricing formulas, the Black-Scholes and the Jarrow-Rudd, are the most accurate ones with Jarrow-Rudd's performance a little better. Among the numerical models, the relaxed trees perform the best, with more than 3 times the forecasting errors of the Black-Scholes and the Jarrow-Rudd. Again, the restricted trees are less accurate than the relaxed trees. The poorest one is the restricted trinomial tree model, whose forecasting error is more than 5 times that of the Black-Scholes'.

Wheat shows only small difference among the methods (Figure 3). The relaxed binomial and trinomial trees as well as the Jarrow-Rudd model are about the same — more accurate than the Black-Scholes, but not very much. The restricted binomial and trinomial trees are less accurate than the Black-Scholes, but not very much either. The least accurate model is still the restricted trinomial tree. The descriptive statistics for the forecasting errors of the six option pricing models on the three commodities are summarized in the table.

Conclusion

The inaccuracy of the Black-Scholes formula arises partly from its being a European option pricing formula while most options traded in real markets are American. The inaccuracy also arises from the assumption that the underlying asset price follows a lognormal distribution. The binomial and the trinomial trees can handle American options while the Edgeworth series provides a possibility to approximate the underlying distribution more accurately than the Black-Scholes formula. For any potential improvement of accuracy of an option-pricing model to be realized, it must be simple enough that people will use it. Due to the difficulty of the parameter determination, the original multi-parameters in the binomial and trinomial trees were reduced to a single volatility parameter by adding artificial constraints on the parameters. Due to this difficulty, the Jarrow-Rudd approximate option pricing formula, a model based on the Edgeworth series, is rarely used in practice.

This study implies the parameters in all of the option pricing formulas. Different from the way that the binomial and trinomial trees are usually used, the artificial constraints are relaxed in this study. Relaxing the restrictions is equivalent to letting skewness and kurtosis to be implied. The initial empirical results justify the idea of relaxing the restrictions. Using the Black-Scholes formula as a comparison, the study results show that the multi-parameter models are always more accurate than their single parameter counterparts. In case of the analytic pricing formulas, the multi-parameter model, the Jarrow-Rudd model, is always more accurate than the single parameter counterpart, the Black-Scholes formula. On the other hand, among the numerical models used in this study, the multi-parameter models, the relaxed binomial and trinomial trees, are always more accurate than the single parameter models, the restricted binomial and trinomial trees.

The relaxed binomial and trinomial trees can be more or less accurate than the Black-Scholes formula, depending on the commodity under study. The relationship between the model performance and the properties of a given commodity is still not clear. However, in practice, if empirical experience consistently show the same pattern for some given commodity, it is rational to use a model that performs significantly well, even though the reason for the significance is not yet clear. The poor performance of the tree models for soybeans is likely due to the American option model not fitting these data. In future research, we will explore if this is due to stale prices being filled in with the Black-Scholes model.

Among the binomial and trinomial trees, the relaxed binomial tree always performs at least as accurate as the relaxed trinomial tree. In view of the fact that the binomial tree is simpler than the trinomial tree, the former should be more favorable than the latter. Since the restricted binomial and trinomial trees are always the two poorest among the six models examined in this study, they should never be used to imply the volatility parameter. Such results are consistent with intuition. Since the parameters in the models are artificially restricted, there must be some real relationship among the parameters being violated. Since more artificial constraints are added to the restricted trinomial tree, its forecasting performance is worse than the less restricted one, the restricted binomial tree model.

Though the results of this paper need to be verified by further studies based on more data, its practical significance is clear: it is possible to use multiple implied parameters to increase the accuracy of currently used option pricing models, especially the Black-Scholes formula.

Figure 1. Forecasting Error Comparison for 1998 July Corn Contract (Black-Scholes = 1)

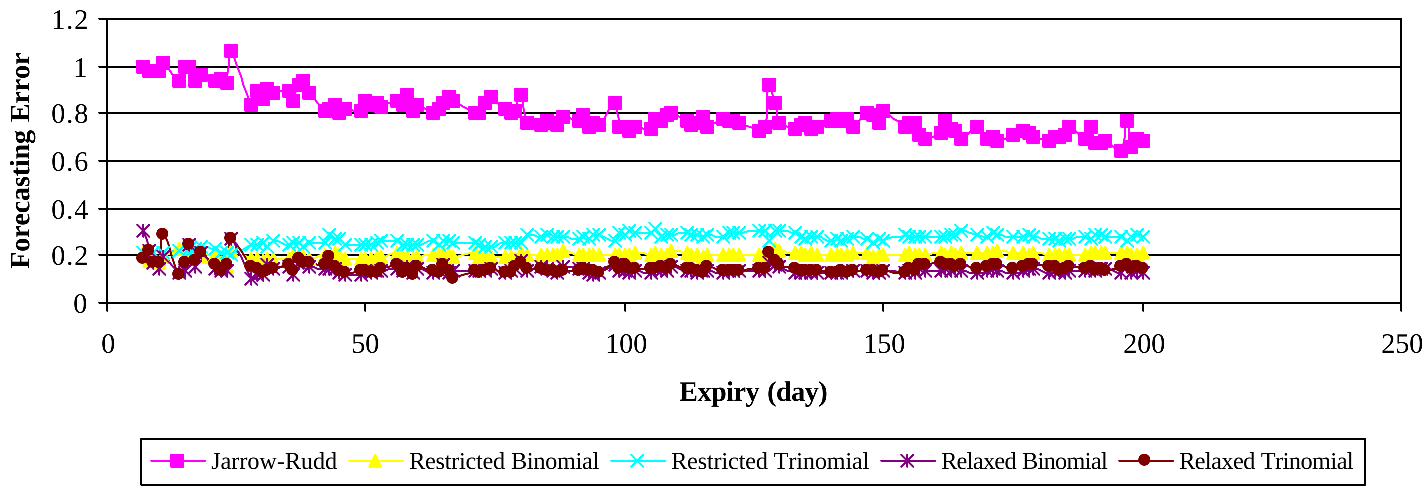


Figure 2. Forecasting Error Comparison for 1998 July Soybean Contract (Black-Scholes = 1)

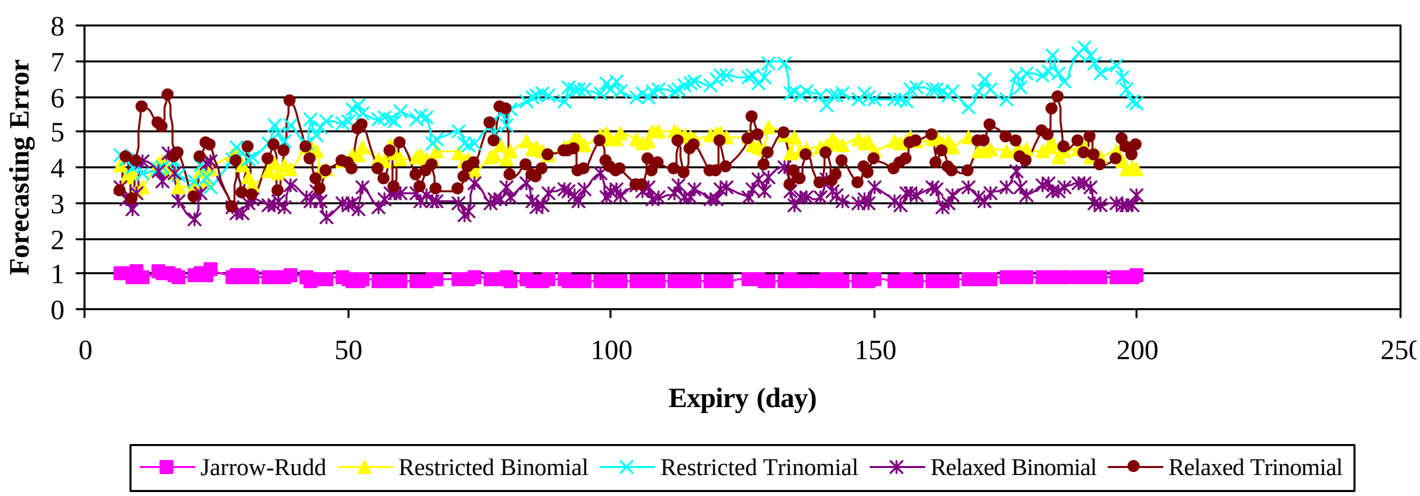


Figure 3. Forecasting Error Comparison for 1998 July Wheat Contract
(Black-Scholes = 1)

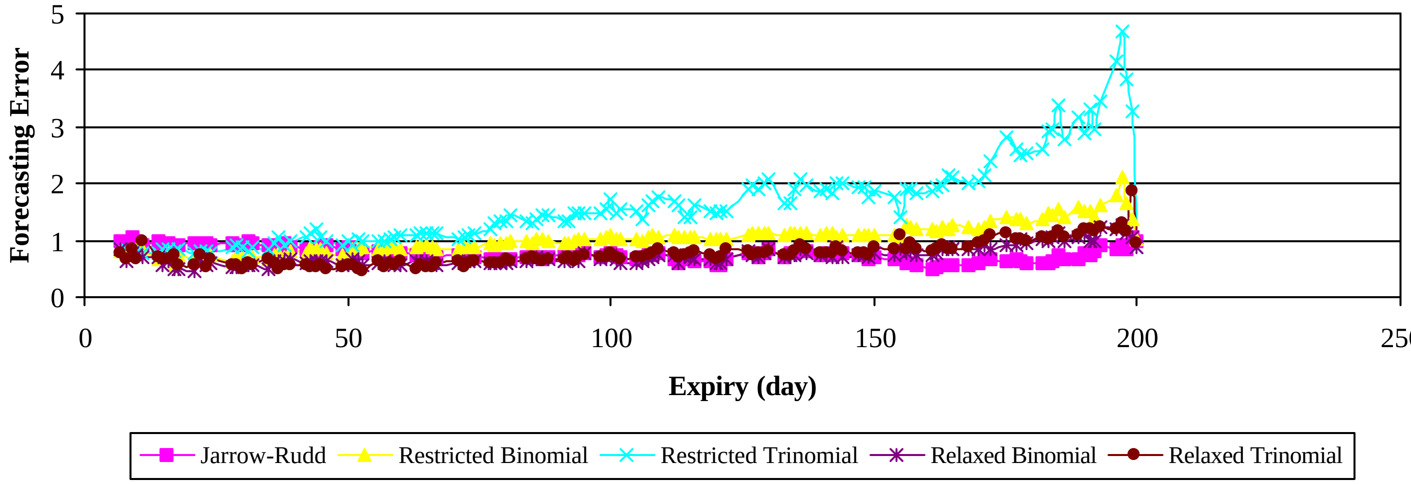


Table 1. Descriptive Statistics of Forecasting Errors of Various Models (%)

Commodity		Model: Black-Scholes	Jarrow-Rudd	Restricted Binomial Tree	Restricted Trinomial Tree	Relaxed Binomial Tree	Relaxed Trinomial Tree
Corn	Mean	4.67	3.58	0.96	1.28	0.64	0.68
	Std	2.08	1.46	0.44	0.61	0.28	0.31
Soybean	Mean	0.84	0.70	3.80	4.99	2.70	3.57
	Std	0.32	0.25	1.60	2.23	1.07	1.42
Wheat	Mean	2.87	2.06	3.11	4.96	2.13	2.26
	Std	1.28	0.83	1.59	2.86	1.05	1.19

Note: The forecasting error is defined as $\frac{\sum_{i=1}^{N} | Call_{ti} - c(q_{t-1}) | + \sum_{j=1}^{M} | Put_{tj} - p(q_{t-1}) |}{\sum_{i=1}^{N} Call_{ti} + \sum_{j=1}^{M} Put_{tj}}$, where $Call_{ti}$ is the premium for the i-th call option contract at the t-th day, Put_{tj} is the premium for the j-th put option contract at the t-th day, c(θ) and p(θ) are the pricing formulas for call and put options respectively with θ as the parameter vector in the formulas.

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